









# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



## THESIS

SEQUENTIAL ESTIMATION OF OPTIMAL AGE  
REPLACEMENT POLICIES WHEN  
DISTRIBUTION OF LIFETIMES IS PHASE TYPE

by

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Sequential Estimation of Optimal Age Replacement Policies  
When Distribution of lifetimes is Phase Type

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## ABSTRACT

Optimal age replacement policies are designed to cut down system failures and minimize maintenance cost. By scheduling planned replacements, a system is replaced at age  $\phi^*$  or at failure, whichever comes first, and the cost of replacement before failure (planned) is less than the cost after failure (unplanned). In this thesis, the distribution of lifetimes is a known, increasing failure rate phase type distribution. To find the optimal age of replacement, the parameters of the underlying phase type distribution must be estimated.

An optimal age sequential estimation procedure is developed. In particular, the phase type distributions parameters are estimated using a matching moments nonlinear programming approach. Since there are many parameters associated with phase type distributions and the distributions include matrix exponential terms, the parameters are in general difficult to estimate. A specific case where the phase type distribution has initial probability vector  $\alpha=(1,0,0)$  is studied for different sample sizes and compared with a similar nonparametric procedure.

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## I. INTRODUCTION

Normally a system or component is replaced when it fails and  $C_1$ , the cost of replacement before failure (planned replacement), is often less than  $C_2$ , the cost after failure (unplanned replacement). The problem of determining when to repair and when to replace failing systems is the concern of management resources. Inefficient management due to the use of nonoptimal maintenance policies can lead to significant system maintenance cost. In general, optimal maintenance policies are designed to cut down the number of system failures and minimize maintenance costs. In this thesis, we study the problem of estimating a type of optimal maintenance policy (the optimal age replacement policy) when the underlying system life distribution is phase-type. In particular, we consider an adaptive estimation scheme where the estimated optimal replacement age is updated each time the system is replaced.

Maintenance is defined to be all activities taken to keep the system in serviceable condition or to bring it back to serviceability. There are two types of maintenance, corrective and preventive maintenance [Ref. 5: pp. 1-8]. Corrective maintenance is used after a failure. This does not necessarily mean that such action has not been foreseen. Preventive maintenance aims to reduce the probability of failure and includes:

- Planned (scheduled) maintenance in which specified components are replaced (usually at regular intervals). The maintenance time is usually based on component lifetime distributions.

-Unplanned (condition-based) maintenance in which the decision to replace or not to replace is made according to the outcome of a diagnostic study.

The policies that are designed to reduce the number or the probability of system failure and maintenance costs are called maintenance policies [Ref. 21: pp. 1-3]. We concentrate on age replacement policies where a system is replaced at age  $T$  or failure, whichever comes first. For this problem to make sense,  $C_1$  must be less than  $C_2$  and the system must age with time i.e. the failure rate of the system must be increasing with age. It is not wise to replace the equipment before failure when the system has a constant or decreasing failure rate such as when the system failures occur according to an exponential distribution. In this case, replacement will not reduce the probability of system failure occurring in the next instance of time. When using an age replacement policy the question always asked is, "At what age should the equipment be replaced". If the replacement occurs too frequently the maintenance costs will be high. If replacement is too infrequent, the system will fail more often than necessary and again, the maintenance cost will be too high. Thus it is desirable to find an "optimal" replacement age. Here, the optimal age is defined as the age which yields the minimum long run expected maintenance costs per unit time.

We will assume that the maintenance action returns the system to the good-as-new condition, thus the same services are provided as before replacement. To accomplish this we assume that the systems used for replacement have lifetimes that are independent and identically distributed according to a phase-type distribution. The class of phase type distributions is large and includes Exponential and Gamma distributions along with



convolutions and mixtures of these distributions. This fact, along with the fact that phase type distributions have a physical interpretation make them particularly well suited for modeling system lifetimes.

The optimal replacement age depends on the underlying phase type distribution. This is in general unknown and must be estimated. Estimation for phase type distributions based on iid lifetimes has been studied by Neuts and Meier (1980). Estimation of the optimal age of replacement for phase type distributions is new. However both parametric and nonparametric estimation based on iid observations have been considered by Arunkumar (1972), Ingram and Schaeffer (1976), Bergman (1977), and Barlow (1978). Here we use a sequential approach for estimating the optimal replacement age. At each replacement the estimate is updated and the new system is subject to an age replacement policy based on the optimal age estimated so far. This type of sequential approach has been studied parametrically by Oclay (1990) and nonparametricly by Bather (1977), Frees and Ruppert (1985), Aras and Whitaker (1991), Wu (1990) and in a decision theoretic framework by Glazebrook, Bailey and Whitaker (1991).

Phase type distributions are described in Chapter II. The sequential estimation procedure to estimate the optimal age of replacement is given in Chapter III, and in Chapter IV we present results comparing the sequential estimation procedure assuming an underlying phase type distribution with a nonparametric procedure. Conclusion and recommendation are given in Chapter V.

## II. PHASE TYPE DISTRIBUTION

### A. GENERAL

A phase type distribution is defined as the distribution of the time until absorption in a finite state Markov process. To determine the distribution of the absorption time, consider an  $m+1$  state, continuous time Markov chain whose infinitesimal generator  $Q$  has the form

$$Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$$

where  $T$  is a nonsingular  $m \times m$  matrix with negative diagonal elements and nonnegative off-diagonal elements,  $T^0$  is vector of length  $m$  with nonnegative elements and  $0 = (0, 0, \dots, 0)$ . Moreover,

$$Te + T^0 = 0'$$

where  $e = (1, 1, \dots, 1)'$ .

By the construction of  $Q$  the state  $m+1$  is absorbing and all other states are transient. The necessary and sufficient condition for this is that the inverse  $T^{-1}$  exists [Ref. 15: pp. 446]. Let the initial probability vector of the Markov chain be given by  $(\alpha, \alpha_{m+1})$  with  $\alpha e + \alpha_{m+1} = 1$  and  $\alpha$  being the  $m$  dimensional initial probability vector of transient states such that  $0 < \alpha e \leq 1$ . Let  $X$  be the time until absorption into the  $(m+1)^{\text{st}}$  state. The probability distribution  $F(x)$  of the time until absorption in the state  $m+1$  corresponding to the initial probability vector  $(\alpha, \alpha_{m+1})$  is given by

$$F(x) = 1 - \alpha \exp(Tx)e, \quad \text{for } x \geq 0, \quad (2.1)$$

where  $\exp(A)$  is the matrix exponential of the matrix  $A$  defined in Ref. 8.

The probability distribution  $F(x)$  on  $[0, \infty)$  is said to be of phase type (PH-distribution) and the pair  $(\alpha, T)$  is called a representation of  $F(x)$ . If  $\alpha_{m+1} > 0$  the distribution  $F(x)$  has a jump of height  $\alpha_{m+1}$  at  $x = 0$ .

On  $(0, \infty)$ , the probability mass is distributed according to a density given by

$$f(x) = \alpha \exp(Tx)T^0, \quad \text{for } x > 0, \quad (2.2)$$

The noncentral moments  $\mu_i = E[X^i]$  of  $F(x)$  are all finite and given by

$$\mu_i = (-1)^i i! (\alpha T^i e), \quad \text{for } i \geq 1 \text{ [Ref. 15: pp. 446]}. \quad (2.3)$$

## B. COST FUNCTION

Let  $X_1, X_2, \dots$  be a sequence of independent and identically distributed (iid) positive random variables with distribution function  $F$ . The sequence  $X_1, X_2, \dots$  represents the sequence of system lifetimes that would be observable if the system were replaced at failure. Let  $C_1, C_2$  ( $C_2 > C_1$ ) and  $\phi$  be the respective cost of planned replacement; unplanned replacement and the replacement age. The long run expected cost per unit time can be verified [Ref. 1: pp. 87] to be

$$C(\phi) = \frac{C_2 F(\phi) + C_1 \bar{F}(\phi)}{\int_0^\phi \bar{F}(u) du} \quad (2.4)$$

where  $\bar{F}(u) = 1 - F(u)$  is the survival function. A sufficient condition that guarantees a unique and finite  $\phi^*$  that minimizes  $C(\phi)$ , is that  $F$  have strictly increasing failure rate.

When the distribution  $F$  is phase type with representation  $(\alpha, T)$ , the long run expected cost function is

$$C(\phi) = \frac{C_2 [1 - \alpha \exp(T\phi) \mathbf{e}] + C_1 [\alpha \exp(T\phi) \mathbf{e}]}{\int_0^\phi \alpha \exp(Tu) \mathbf{e} du} \quad (2.5)$$

$$= \frac{C_2 - \alpha \exp(T\phi) \mathbf{e} [C_2 - C_1]}{\alpha T^{-1} [\exp(T\phi) - I] \mathbf{e}} \quad (2.6)$$

where  $I$  is an identity matrix.

The phase type distribution does not necessarily have increasing failure rate, so the cost function  $C(\phi)$  does not necessarily have a unique finite minimum. In the  $m=3$  case even when the initial probability of the transient states is fixed the minimum can be unique and finite or can occur at infinity as shown in Figure 1, where  $\alpha = (1, 0, 0)$  and

$$\mathbf{T}_1 = \begin{bmatrix} -15 & 13 & 2 \\ 3 & -15 & 12 \\ 2 & 5 & -15 \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} -20 & 10 & 10 \\ 10 & -20 & 10 \\ 10 & 10 & -25 \end{bmatrix}.$$

In Figure 2, the same phenomena is shown when  $F$  has the same nonsingular  $3 \times 3$  matrix

$$\mathbf{T}_3 = \begin{bmatrix} -18 & 13 & 5 \\ 10 & -25 & 15 \\ 4 & 4 & -24 \end{bmatrix}$$

but the initial probabilities are different.

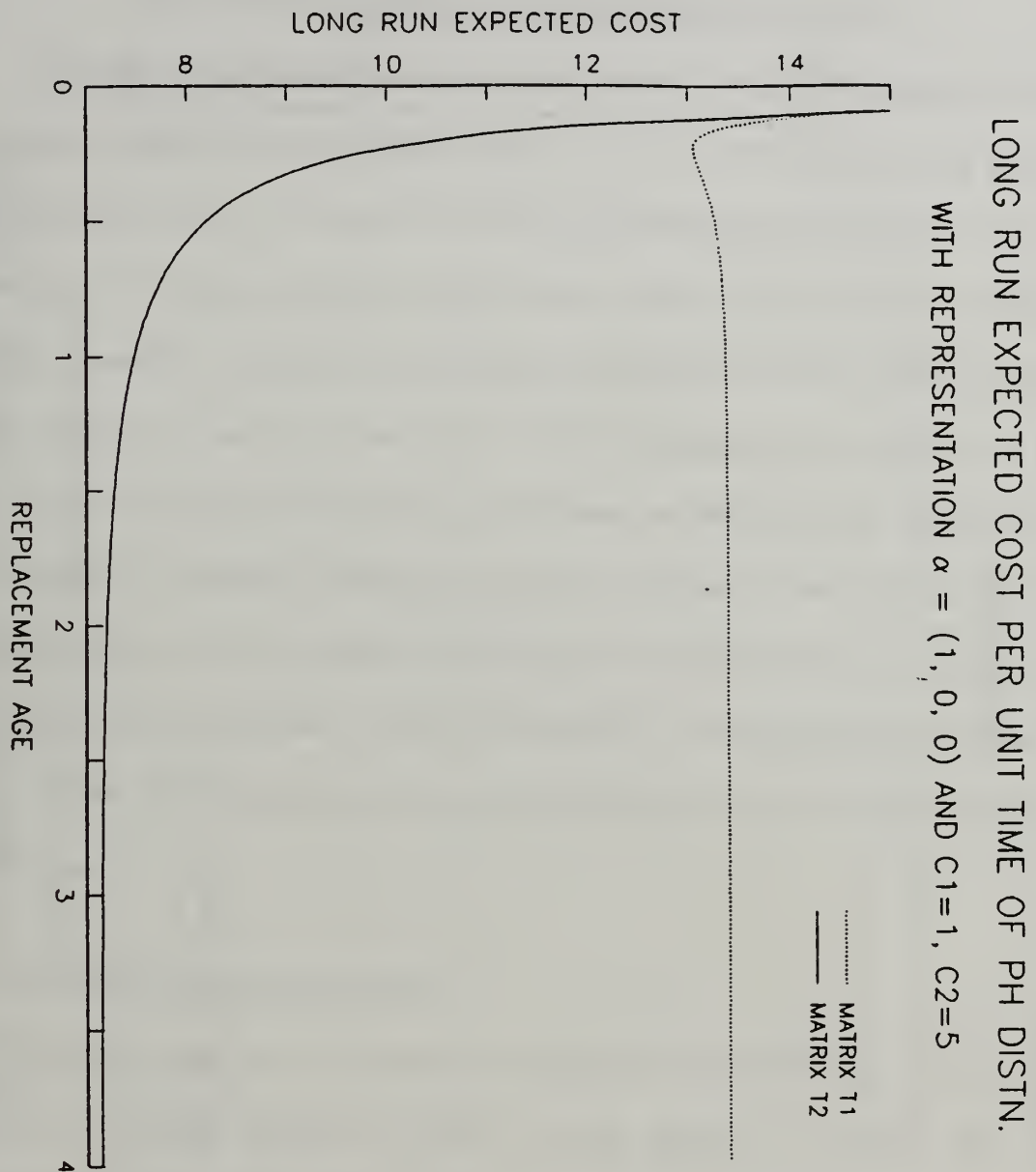
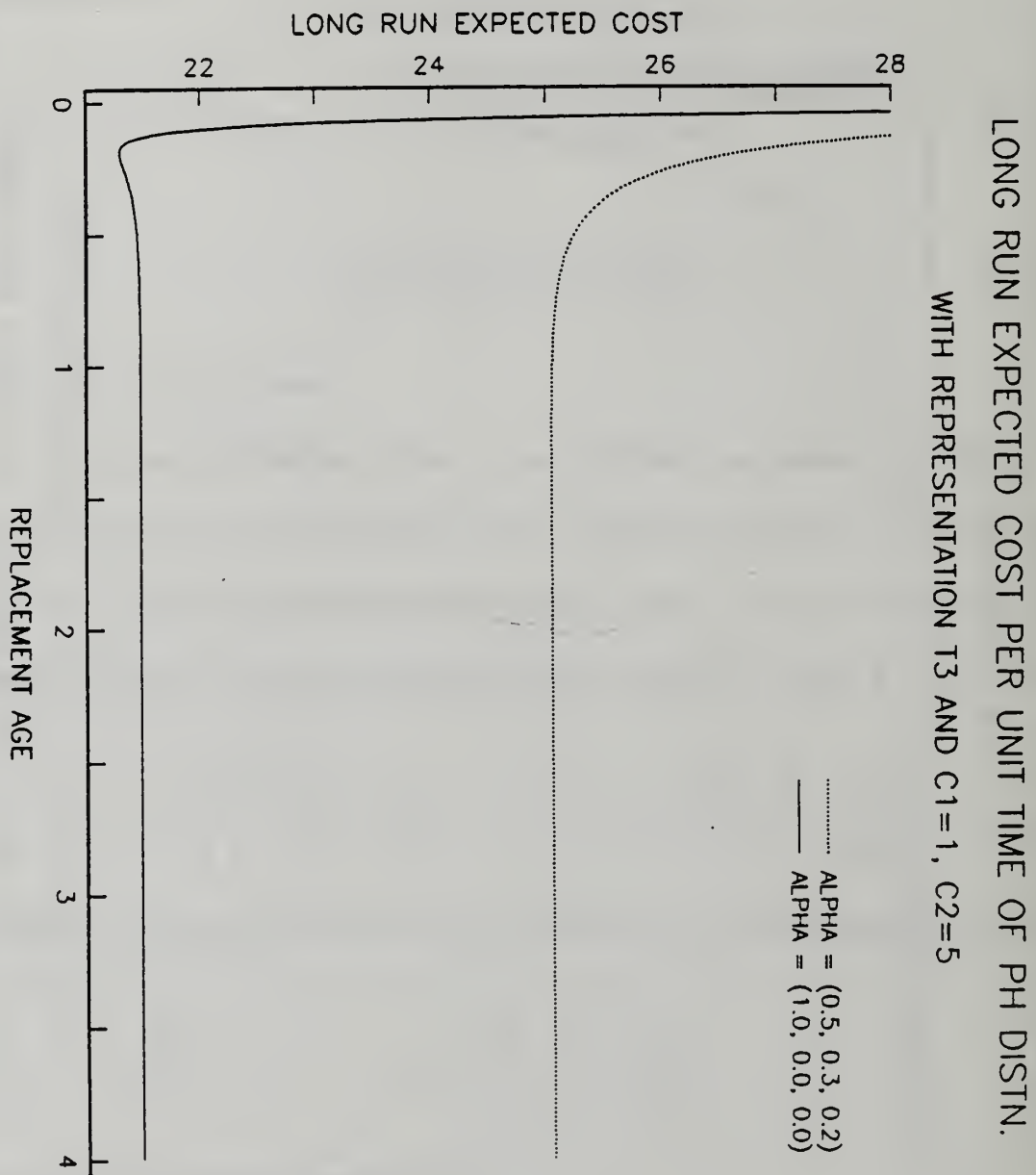


Figure 1 The long run expected cost per unit time curves of PH distribution with the same  $\alpha$  and different  $T = T_1$  or  $T_2$



**Figure 2** The long run expected cost per unit time curves of PH distribution with the same  $T$  and different  $\alpha$



### III. THE SEQUENTIAL ESTIMATION PROCEDURE

In theory, any distribution of a nonnegative random variable can be approximated arbitrarily closely by a PH distribution [Ref. 11: pp. 1-2]. This means that the PH distributions are dense in  $\mathcal{F}$  where  $\mathcal{F}$  is the set of distributions with support on  $[0, \infty)$ . The paper of Johnson and Taaffe (1988) shows that the finite mixtures of Erlang distributions and PH distributions with a Coxian representation are both dense in  $\mathcal{F}$ . Since both families are subsets of the family of PH distributions, this implies that PH distribution are dense in  $\mathcal{F}$  [Ref. 10: pp. 1-8]. Thus, the PH distribution can be used to approximate any unknown distribution of lifetimes (lifetime is always greater than or equal 0). Another feature that makes PH distributions a desirable choice to model system lifetimes is that they may have a physical interpretation. The absorbing state represents system failure and the transient states may represent different levels of a system's ability to function.

#### A. PARAMETER ESTIMATION

There are many ways to estimate the parameters of a distribution including the method of maximum likelihood (MLE), and the method of moments. The PH distributions have special properties which make estimation difficult. First, the number of parameters is not fixed, it varies with the number of transient states, e.g. a PH distribution with 2 transient states has 6 parameters, a PH distribution with 3 transient states has 12 parameters etc. Also the probability distribution and density function include

the exponential of a matrix. This makes it hard to work with the likelihood function. In this thesis, the moment matching method is used to estimate the parameters of a phase type distribution but the MLE method and its associated problems are also discussed . In addition, the number of transient states,  $m$ , is assumed to be known.

## 1. The Maximum Likelihood Method

Let  $X_1, X_2, \dots$  represent the sequence of system lifetimes, where  $X_1, X_2, \dots$  are iid PH distributions with representation  $(\alpha, T)$ . After  $N$  observations the data available for estimation will be  $(Z_1, \delta_1), (Z_2, \delta_2), \dots, (Z_N, \delta_N)$  where  $Z_i = \min(X_i, \hat{\phi}_{i-1}^*)$ ,  $X_i$  is the  $i^{\text{th}}$  lifetime and  $\hat{\phi}_{i-1}^*$  is the optimal age replacement estimated after  $i-1$  observations and  $\delta_i = 1$  if  $X_i \leq \hat{\phi}_{i-1}^*$  and  $\delta_i = 0$  otherwise. Assuming system lifetimes have a PH distribution the likelihood for the replacement ages  $Z_1, \dots, Z_N$  and types of replacement  $\delta_1, \delta_2, \dots, \delta_N$  is

$$\begin{aligned} L(\alpha, T) &= \prod_{i=1}^N [\alpha \exp(TZ_i) T^*]^{\delta_i} [\alpha \exp(TZ_i) \mathbf{e}]^{1-\delta_i} \\ &= \prod_{i=1}^N [-\alpha \exp(TZ_i) T\mathbf{e}]^{\delta_i} [\alpha \exp(TZ_i) \mathbf{e}]^{1-\delta_i} . \end{aligned}$$

It is too difficult to maximize this likelihood directly by differentiating  $L(\alpha, T)$  and solving for the MLE's. There are several reasons for this. The likelihood and its derivatives include the exponential of a matrix form that cannot be written in closed form and the parameters are subject to numerous constraints. In addition, there are many likelihood equations due to the number of parameters (3 transient states will have up to

12 equations). An alternate approach is to find approximate MLE's using nonlinear programming algorithms i.e.:

$$\begin{aligned}
 \text{MAX } L(\alpha, T) &= \prod_{i=1}^N [-\alpha \exp(TZ_i) T e]^{\delta_i} [\alpha \exp(TZ_i) e]^{1-\delta_i} \\
 \text{S.T. } \quad \alpha &\geq 0 \\
 \alpha e &= 1 \\
 t_{ii} &< 0 & \text{for } i = 1, 2, \dots, m \\
 t_{ij} &\geq 0 & \text{for } i \neq j \\
 \sum_{i \neq j} t_{ij} &\leq -t_{ii} & \text{for } i = 1, 2, \dots, m \\
 \det(T) &\neq 0
 \end{aligned}$$

where  $m$  = the number of transient states of the PH distribution. Even when  $m$  is assumed to be known, there is still the problem of approximating the exponential of a matrix and the optimization software to take care of this problem is not available. Thus, it is very difficult to use this approach to estimate the PH distribution parameters.

## 2. The Moment Matching Method.

The PH distribution is a complicated distribution, there are many parameters and there are difficulties with computing the exponential of a matrix. One property, the existence of  $T^{-1}$ , implies that all noncentral moments of a PH distribution are finite. The  $k^{\text{th}}$  moment is given by

$$\mu_k = (-1)^k k! \alpha T^{-1} e. \quad (3.1)$$

Moment matching is a common method for estimating. Using moment matching bypasses the problem of evaluating the exponential of a matrix [Ref. 12: pp.3].

Let  $m$  be the number of transient states of the PH distribution. There are  $m(m+1)$  parameters that need to be estimated, thus the  $m(m+1)$  equations from the moments that need to be solved are

$$\begin{aligned}\hat{\mu}_1 &= -\alpha T^{-1} \mathbf{e} \\ \hat{\mu}_2 &= 2! \alpha T^{-1} \mathbf{e} \\ &\cdot \\ &\cdot \\ &\cdot \\ \hat{\mu}_{m(m+1)} &= (-1)^{m(m+1)} (m^2 + m)! \alpha T^{-1} \mathbf{e},\end{aligned}$$

where  $\hat{\mu}_k$  is the  $k^{\text{th}}$  sample moment.

This method is still inappropriate due to the large number of equations and the fact that when the dimension of matrix  $T$  is big and the moments are of high order, substantial amount of error is introduced when trying to solve these equations. Johnson and Taaffe have shown that the moment matching method and nonlinear programming approaches can be used to estimate the parameters of a PH distribution. They match only the second and third standardized moments [Ref. 11: pp. 3-11].

Let  $\bar{\mu}_i$  be the  $i^{\text{th}}$  central moment. The second standardized moment, the coefficient of variation (C), defined as

$$C = \frac{\text{standard deviation}}{\text{mean}},$$

can be written as

$$C = \frac{(\bar{\mu}_2)^{1/2}}{\mu_1} . \quad (3.2)$$

The third standardized moment, the coefficient of skewness, ( $\gamma$ ) is defined as

$$\gamma = \frac{\bar{\mu}_3}{(\bar{\mu}_2)^{3/2}} . \quad (3.3)$$

And the  $t^{\text{th}}$  standardized moment is  $\bar{\mu}_t/(\bar{\mu}_2)^{t/2}$  for  $t = 3, 4, \dots$ .

The reason for matching standardized moments is that the standardized moments do not change with scale changes. So, the shape of a PH distribution with representation  $(\alpha, T)$  will not change if it is multiplied by a nonnegative number.

The nonlinear programming formulation for matching  $C$  and  $\gamma$  to a continuous PH distribution is given by

$$\text{MIN.} \quad w_1(C-C(0))^2 + w_2 (\gamma-\gamma(0))^2$$

S.T.

$$\alpha \geq 0$$

$$\alpha e = 1$$

$$t_{ii} < 0 \quad \text{for } i = 1, 2, \dots, m$$

$$t_{ij} \geq 0 \quad \text{for } i \neq j$$

$$\sum_{i \neq j} t_{ij} \leq -t_{ii} \quad \text{for } i = 1, 2, \dots, m$$

$$\det(T) \neq 0$$

where

$C(0)$  = the second standardized moment of the current solution.

$\gamma(0)$  = the third standardized moment of the current solution.



$m$  = the number of transient states of the PH distribution.

$t_{ij}$  = the elements of row  $i$  and column  $j$  of nonsingular matrix  $T$ .

$w_1, w_2$  are positive weights chosen to guide the search and  $w_1 \geq 3w_2$ .

The moments are matched when the objective function value is zero. The solutions which match the target moments are called "moment-matching solutions" [Ref. 11: pp. 5].

Even with fixed dimension, this procedure may have many solutions and these solutions are unpredictable. There are some practical points that need to be made associated with the properties of PH distribution and the problem of using nonlinear programming to get the moment matching solutions:

- Different combinations of initial solutions and algorithm parameters may lead to different moment-matching solutions. Also, some combinations of initial solution and algorithm parameters may not lead to a moment-matching solution and the nonlinear programming algorithm may not converge.

- For a given dimension we do not know how many solutions exist or how to guide the search toward a preferable solution.

- The moment-matching solutions do not guarantee that the estimated cost function will have a finite minimum. When choosing among several solutions, a solution which gave a cost function with a finite minimum was always chosen.

- The values of  $w_1$  and  $w_2$  can be modified to guide the search; generally  $w_1 \geq 1$ ,  $w_2 \geq 1$  and  $w_1 \geq 3w_2$ . The magnitude of  $w_1$  and  $w_2$  can be increased to get more or sufficient accuracy [Ref. 11: pp. 6].



- If the feasible solutions are known, parameters can be bounded to guide the search. However, some bounds on the parameters may not lead to a moment-matching solution or to a solution whose cost function does not have a unique finite minimum.

- User interaction is often necessary in obtaining the appropriate or preferable solutions.

## B. THE SEQUENTIAL ESTIMATION PROCEDURE

Let  $X_1, X_2, \dots, X_N$  be the sequence of lifetimes of the system and  $\{\hat{\phi}_N^*\}$  be the sequence of estimators of  $\phi^*$  where  $\hat{\phi}_N^*$  is the estimator after  $N$  replacements. Then after  $N$  observations, we have the right censored data  $(Z_1, \delta_1), (Z_2, \delta_2), \dots, (Z_N, \delta_N)$ .

where  $Z_i = \min (X_i, \hat{\phi}_{i-1}^*)$

$$\delta_i = 1 \quad \text{if } X_i \leq \hat{\phi}_{i-1}^* \quad \text{otherwise } \delta_i = 0 .$$

Because the data are right censored the usual method of moments approach for estimation needs to be modified. Rather than use sample moments calculated from an empirical distribution, we use moments from a nonparametric estimator of  $F$  based on the censored data. From the right censored data after  $N$  observations, the procedure to compute the estimators  $\{\hat{\phi}_i^*\}$  is developed as follows.

1. Use the right censored data sequence  $(Z_1, \delta_1), (Z_2, \delta_2), \dots, (Z_N, \delta_N)$  to estimate moments with  $\bar{F} = 1 - F$  by the product-limit estimator:

$$\hat{\bar{F}}(t) = \prod_{(i, Z_{(i)} \leq t)} \left( \frac{N-i}{N-i+1} \right)^{\delta_{(i)}} \quad (3.4)$$

where  $Z_{(1)}, Z_{(2)}, \dots, Z_{(N)}$  are the order statistics of  $Z_1, Z_2, \dots, Z_N$  and  $\delta_{(1)}, \delta_{(2)}, \dots, \delta_{(N)}$  are ordered according to the ordering of  $Z_{(1)}, Z_{(2)}, \dots, Z_{(N)}$ . Then calculate probabilities  $P_i$ , associated with  $Z_{(i)}$ , by

$$P_i = \hat{\bar{F}}(Z_{(i)}^-) - \hat{\bar{F}}(Z_{(i)}) . \quad (3.5)$$

Note, since  $\bar{F}$  has discontinuities only at  $Z_{(i)}$  where  $\delta_{(i)} = 1$ , that  $P_i = 0$  when  $\delta_{(i)} = 0$ . Then estimates of the  $k^{\text{th}}$  moment are given by

$$\hat{E}[X^k] = \sum_{i=1}^N P_i Z_{(i)}^k \quad (3.6)$$

The estimate of the 2<sup>nd</sup> standardized moment is

$$\hat{C} = \frac{\sqrt{\hat{E}[X^2] - (\hat{E}[X])^2}}{\hat{E}[X]} \quad (3.7)$$

and the estimate of the 3<sup>rd</sup> standardized moment is

$$\hat{\gamma} = \frac{\hat{E}[X^3] - 3\hat{E}[X]\hat{E}[X^2] + 2(\hat{E}[X])^3}{(\sqrt{\hat{E}[X^2] - (\hat{E}[X])^2})^3} . \quad (3.8)$$

2. Estimate parameters of PH distribution  $(\alpha, T)$ , by matching the second and third standardized moments using the nonlinear programming approach.

When executing the nonlinear programming algorithm using the model described in the previous section, the  $\det(T) \neq 0$  constraint is replaced by a constraint on the expected lifetime. The reason for doing this is that we could not formulate an algebraic

constraint equivalent to the constraint  $\det(\mathbf{T}) \neq 0$ . The problem can be solved by a constraint on the expected lifetime, i.e. by taking

$$\alpha \mathbf{T}^{-1} \mathbf{e} = \hat{E}[X] \quad (3.9)$$

where  $\hat{E}[X]$  is calculated in step 1. This constraint will make the search for preferable solutions easier. The initial solution is chosen by the user. Some initial solutions may or may not lead to a moment matching solution. Typically, the initial vector  $\alpha$  is  $(1/m, 1/m, \dots, 1/m)$  or  $(1, 0, \dots, 0)$  and an initial matrix  $\mathbf{T}$  consists of  $t_{i,i} = -1$  and  $t_{i,j} = 1/m$  for  $i \neq j$ ,  $i = 1, 2, \dots, m$ . In this thesis the initial vector  $\alpha$  is  $(1, 0, \dots, 0)$  and an initial matrix  $\mathbf{T}$  is taken to have  $t_{i,i} = -1$  and  $t_{i,j} = 1/m$  [Ref. 11: pp. 9] .

3. Using the parameters estimated in step 2, the estimated cost function is

$$\hat{C}(\phi) = \frac{C_2 - \hat{\alpha} \exp(\hat{\mathbf{T}}\phi) \mathbf{e} [C_2 - C_1]}{\hat{\alpha} \hat{\mathbf{T}}^{-1} [\exp(\hat{\mathbf{T}}\phi) - \mathbf{I}] \mathbf{e}} \quad (3.10)$$

The new estimate of the optimal age of replacement  $\hat{\phi}_N^*$  is taken to be the  $\phi$  that minimizes (3.10) by enumerative search.

4. Compare the  $\hat{\phi}_N^*$  with  $X_{N+1}$  then repeat Step 1.

The initial solutions for matrix  $\mathbf{T}$  and vector  $\alpha$  in step 2 are taken to be the previously estimated values of  $\mathbf{T}$  and  $\alpha$ . In case the moment matching solution cannot be met, the initial solution in step 2 will be taken to be the original initial solution and the previous optimal age replacement is used for step 3.

The replacement cost for  $i^{\text{th}}$  system is  $C_2$  if  $X_i < \hat{\phi}_{i-1}^*$ ; otherwise the replacement cost is  $C_1$ . The actual total replacement cost for the first  $N$  systems that are observed is

$$C_N = \sum_{i=1}^N [C_2 \times \delta_i + C_1 \times (1 - \delta_i)] \quad (3.11)$$

and the total operating time for the N systems is

$$t_N = \sum_{i=1}^N \text{Min}(X_i, \hat{\phi}_{i-1}^*) = \sum_{i=1}^N Z_i \quad (3.12)$$

So, the actual average cost (AAC) after N replacement is computed by

$$AAC_N = \frac{C_N}{t_N} \quad (3.13)$$

In this thesis, the moment matching nonlinear programming approach for estimating parameters of PH distributions uses the GAMS program as shown in Appendix A (for a PH distribution with 3 transient states). The Fortran programs are written to estimate the optimal age of replacement, second and third standardized moments and for data preparation as shown in Appendix B.

#### IV.COMPARISON WITH NONPARAMETRIC PROCEDURE

##### A. NONPARAMETRIC PROCEDURE

An alternate approach for estimating  $\phi^*$  is to estimate nonparametrically, as in Aras and Whitaker (1991). The procedure is much simpler computationally than the parametric procedure described in the previous section for PH distributions. In the nonparametric procedure after N replacements the product limit estimator  $\hat{\bar{F}}$  of  $\bar{F}$  given in (3.4) is used to estimate the cost function  $C(\phi)$  as

$$\hat{C}_N(\phi) = \frac{C_2 \hat{F}(\phi) + C_1 \hat{\bar{F}}(\phi)}{\int_0^\phi \hat{\bar{F}}(u) du} \quad (4.1)$$

where  $\hat{F} = 1 - \hat{\bar{F}}$  is the estimator of the cdf  $F$ , the estimator  $\hat{\phi}_N^*$  of  $\phi^*$  is then found by minimizing  $\hat{C}_N(\phi)$ . In general one would expect parametric procedures to do much better than nonparametric procedures. However, with the numerical difficulties involved with estimating parameters for PH distribution and the fact that the family of PH distributions is so large it is not obvious which procedure is best. The criterion for comparison is the actual average cost per unit time,  $AAC_N$ , after N replacements.



## B. COMPARISON OF NONPARAMETRIC AND PARAMETRIC PROCEDURE FOR THE PH DISTRIBUTION

Simulation was used to compare sequential estimation based on PH distributions with nonparametric sequential estimation,  $C_1 = 100$ ,  $C_2 = 500$ . System lifetimes were generated by an increasing failure rate PH distribution with representation  $(T_{41}, \alpha_{41})$  where,

$$T_{41} = \begin{bmatrix} -5 & 4 & 1 \\ 1 & -6 & 5 \\ 1 & 1 & -7 \end{bmatrix}, \quad \alpha_{41} = (1.0, 0.0, 0.0)$$

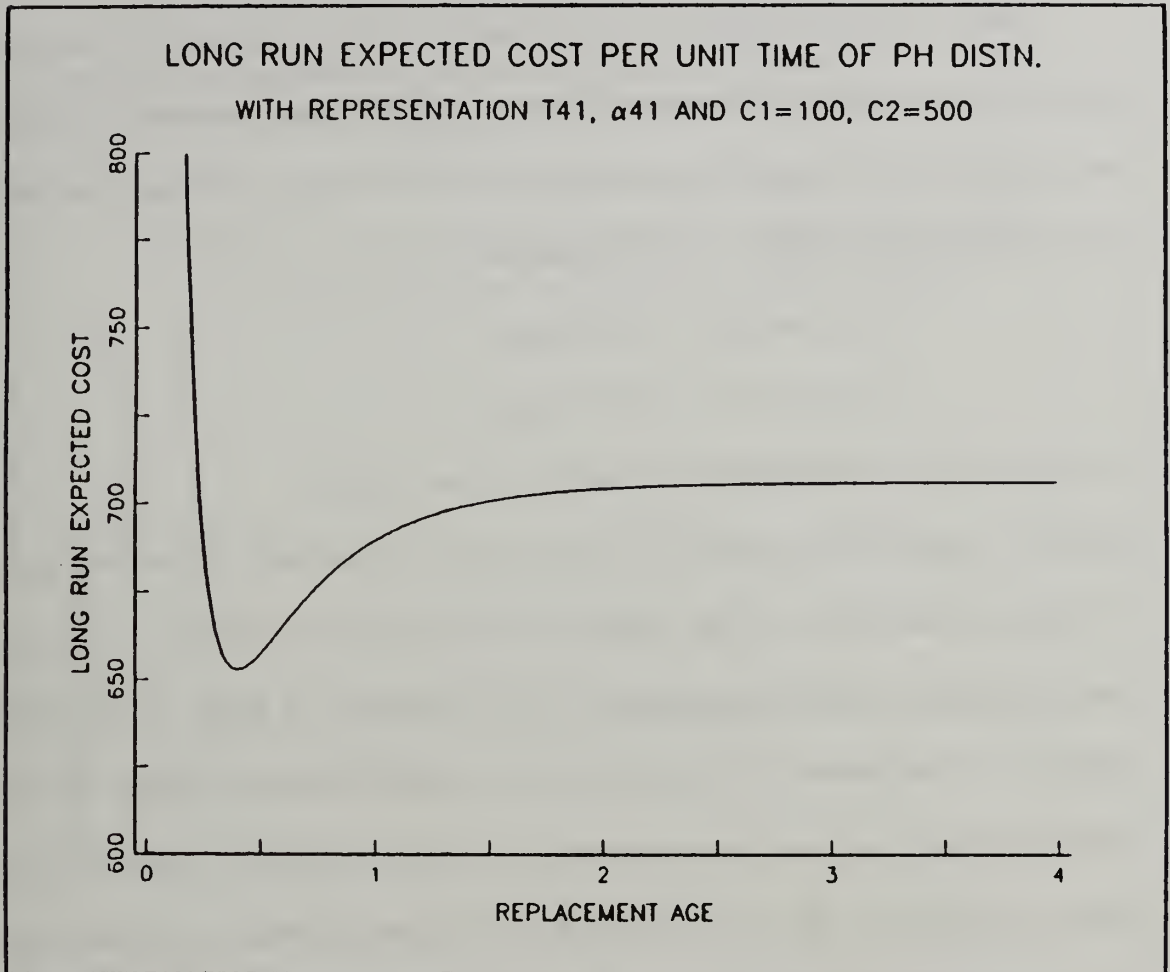
For this representation the long run expected cost per unit time is given in Figure 3.

To give the parametric procedure a chance, the first 25 system lifetimes are uncensored. After the first 25 observations, the parameters of the PH distribution are estimated sequentially as in Chapter III Section B. The actual average costs (AAC) are calculated and compared for small, medium and large sample sizes with planned cost  $C_1 = 100$  and unplanned cost  $C_2 = 500$ . The 20 initial data sets are simulated as shown in the Appendix A.

The programs are separated into 5 parts. The first one is written in GAMS as shown in Appendix A, to estimate the parameters of a PH distribution by the nonlinear programming approach. The rest are written in Fortran as shown in Appendix B, to estimate the optimal age of replacement based on the estimated PH distribution from the GAMS program; to simulate the system lifetime; to prepare the right censored data; to



estimate the expected lifetime, second and third standardized moments; and finally to tabulate the results and to prepare the initial data for the next estimation.



**Figure 3** Long run expected cost per unit time curve of PH distribution with representation  $T_{41}$  and  $\alpha_{41}$

In this simulation, the number of transient states is fixed at 3 and the initial probability vector  $\alpha = (1, 0, 0)$ . By fixing  $\alpha = (1, 0, 0)$  we reduce the number of parameters to be estimated and use a model in which all systems start in the same state, when states are thought of as the level of system repair, this choice better reflects the idea that replacement systems are as good as new. Bounded parameters and user

interaction are used in guiding each estimation during the simulation to a preferable solution.

Tables 1, 2 and 3 summarize the simulation results of the actual average cost resulting from sequential estimation of PH and the nonparametric procedure. Also given are the signed ranks of the differences in the actual average costs. These are used in the Wilcoxon Signed-Rank test to test the hypothesis:

$$H_0: E[AAC_{PH}] = E[AAC_{NONP}]$$

$$H_a: E[AAC_{PH}] < E[AAC_{NONP}]$$

by using  $T^+$  the sum of the positive ranks as the test statistic .

Tables 1, 2 and 3 give the statistic  $T^+$  as 35, 10 and 2 respectively. When compared to the value in Table 9 [Ref. 13: pp. 780] at  $N=20$  all p-values are less than 0.005 which implies that the sequential estimation of a PH distribution is better than sequential estimation of nonparametric for all sample sizes (small, medium and large). The box plots in Figures 5 and 6 of actual average cost versus the number of replacements at different sample sizes show that the actual average costs decrease as the number of replacements increase and that  $AAC_{PH}$  for the parametric procedure decrease more than the  $AAC_{NONP}$  for the nonparametric procedure.

A second, PH distribution was chosen to compare the parametric PH procedure with the nonparametric procedure. The second PH distribution has an average long run expected cost function that is more shallow than the first PH distribution. It has representation  $T_{42}$  and  $\alpha_{42}$

where

$$T_{42} = \begin{bmatrix} -5 & 4 & 1 \\ 1 & -6 & 5 \\ 1 & 2 & -7 \end{bmatrix}, \quad \alpha_{42} = (1.0, 0.0, 0.0),$$

and the long run expected cost per unit time is given in Figure 4.

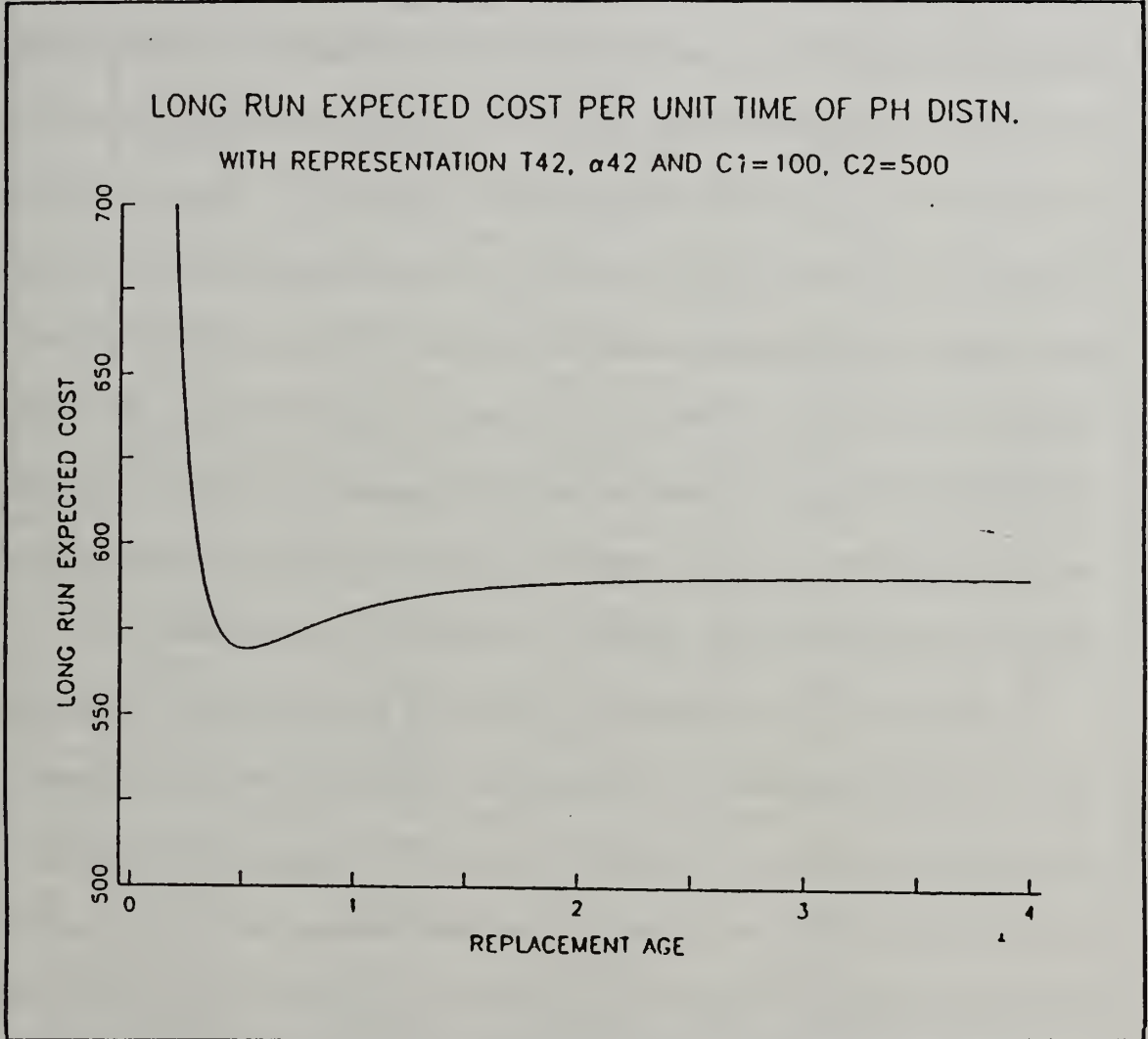


Figure 4 Long run expected cost per unit time curve of PH distribution with representation  $T_{42}$  and  $\alpha_{42}$

Tables 4, 5 and 6 summarize the result and give the statistic  $T^+$  of 93, 81 and 47 respectively. The p-value taken from table 9 [Ref. 13: pp. 780] indicated that the

parametric procedure is not better than the nonparametric procedure for small and moderate sample sizes. In fact, for some cases (e.g. runs 16 and 20 in Table 4 and runs 1, 5 and 16 in Table 5) the actual average cost resulting from using the nonparametric procedure can be quite a bit less than the actual average cost resulting from the parametric procedure. This is due to the fact that  $C(\phi)$  is shallow at  $\phi^*$ , thus even though  $\hat{C}(\phi)$  may be a reasonable estimator of  $C(\phi)$ , the variance of its minimizing value  $\hat{\phi}^*$  is higher than for the previous PH distribution. This means that  $\phi^*$  is often underestimated. Because  $C(\phi)$  increases so rapidly to the left of  $\phi^*$ , underestimating  $\phi^*$  can increase the actual average costs dramatically. Another difficulty is that the estimated  $\hat{C}(\phi)$  may be relatively flat around  $\phi^*$ . This causes numerical difficulties with the nonlinear programming algorithm. For large sample sizes the parametric procedure does do better than the nonparametric procedure. Here  $\hat{C}(\phi)$  estimates  $C(\phi)$  more closely and it is easier for the nonlinear programming algorithm to identify the correct solution.

In Figures 5 and 6, the improvement in actual average cost with sample size is shown for both the parametric and nonparametric procedure for the PH distribution with representation  $T_{41}$  and  $\alpha_{41}$ . As expected, both procedures improve (actual average costs decrease) with sample size. However, the actual average cost for the parametric procedure decreases faster than for the nonparametric procedure. Both procedures exhibit considerable variability. The solutions have got still mixed that we should do more study on another representation until the more precise solutions come up.



**Table 1 COMPARISON OF ACTUAL AVERAGE COSTS FOR SMALL SAMPLE SIZES (N = 50) OF PH DISTRIBUTION WITH  $T_{41}$  AND  $\alpha_{41}$**

Run	AAC <sub>PH</sub>	AAC <sub>NONP</sub>	AAC <sub>RF</sub>	DIFFERENCE	RANK
1	690.095	754.555	759.142	-64.46	16
2	714.728	775.712	780.872	-83.644	14
9	700.342	684.379	688.578	15.963	4 *
4	666.014	725.331	738.987	-59.317	12
5	714.330	757.528	735.058 \$	-43.198	8
8	544.322	621.505	621.507	-77.183	14
7	721.505	810.149	810.149	-88.644	20
8	565.854	617.458	636.556	-51.604	11
9	676.511	725.394	702.676 \$	-48.883	14
10	698.572	766.182	809.629	-67.61	14
11	814.989	767.586	801.150	47.403	9
12	644.921	667.553	672.116	-22.632	8
13	761.516	686.794	679.892 !	74.722	18 *
16	608.477	601.215	617.711	7.262	1 *
15	706.591	767.782	782.198	-61.191	14
16	629.294	667.805	682.719	-38.511	6
17	688.572	755.475	748.353 \$	-66.903	16
16	636.963	624.132	665.344	12.831	3 *
19	583.226	625.161	609.036 \$	-41.935	7
20	644.065	654.691	676.005	-10.626	2

AAC<sub>RF</sub> = Actual average cost replacing when failure occurs

DIFFERENCE = AAC<sub>PH</sub> - AAC<sub>NONP</sub>

\* positive rank

T<sup>+</sup> = 35 ;

p-value < 0.005

\$ AAC<sub>RF</sub> is less than only AAC<sub>NONP</sub>

! AAC<sub>RF</sub> is less than AAC<sub>PH</sub> and AAC<sub>NONP</sub>

**Table 2 COMPARISON OF ACTUAL AVERAGE COSTS FOR MEDIUM SAMPLE SIZES (N = 100) OF PH DISTRIBUTION WITH  $T_{41}$  AND  $\alpha_{41}$**

Run	AAC <sub>PH</sub>	AAC <sub>NONP</sub>	AAC <sub>RF</sub>	DIFFERENCE	RANK
1	648.127	723.170	732.263	-75.043	18
2	690.971	786.554	789.767	-95.583	20
3	649.886	644.710	683.933	5.176	2 *
4	696.858	747.640	755.982	-50.782	11
9	690.195	688.749	682.840 !	1.446	1 *
6	577.742	607.414	610.850	-29.672	5
7	686.414	766.365	762.245 \$	-79.951	17
9	550.106	612.046	638.286	-61.94	14
9	637.268	696.249	689.552 \$	-58.981	12
10	684.253	730.138	786.920	-45.885	16
11	802.039	790.698	807.530	11.341	4 *
12	615.087	687.719	695.174	-72.632	17
13	677.302	666.119	674.830	11.183	3
14	586.838	628.687	638.451	-41.849	8
15	726.213	749.507	737.792 \$	-23.294	5
16	626.605	694.232	667.462 \$	-67.627	16
17	671.908	732.053	758.912	-60.145	13
18	632.954	667.156	727.346	-34.202	7
19	635.989	679.399	667.141 \$	-43.41	9
20	590.881	657.064	701.735	-66.183	15

AAC<sub>RF</sub> = Actual average cost replacing when failure occurs

DIFFERENCE = AAC<sub>PH</sub> - AAC<sub>NONP</sub>

\* positive rank

$T^+ = 10$  ;

p-value < 0.005

\$ AAC<sub>RF</sub> is less than only AAC<sub>NONP</sub>

! AAC<sub>RF</sub> is less than AAC<sub>PH</sub> and AAC<sub>NONP</sub>



**Table 3 COMPARISON OF ACTUAL AVERAGE COSTS FOR LARGE SAMPLE SIZES (N = 200) OF PH DISTRIBUTION WITH  $T_{41}$  AND  $\alpha_{41}$**

Run	AAC <sub>PH</sub>	AAC <sub>NONP</sub>	AAC <sub>RF</sub>	DIFFERENCE	RANK
1	622.560	655.891	655.865 \$	-33.331	7
2	658.640	711.831	708.966 \$	-53.191	11
3	623.823	635.065	671.611	-11.242	4
4	678.141	749.866	754.490	-71.725	14
5	674.548	714.291	707.926 \$	-39.743	9
8	585.435	645.920	647.927	-60.485	13
7	647.302	729.159	733.860	-81.857	20
8	581.711	607.068	649.386	-25.357	5
5	638.465	678.078	719.904	-81.439	19
10	655.709	717.569	766.020	-61.86	14
11	760.478	779.362	762.986 \$	-18.884	4
12	589.088	649.545	681.004	-60.457	18
13	696.081	679.816	712.761	16.265	2 *
18	602.032	643.703	649.832	-47.8	10
15	685.977	767.133	755.331 \$	-81.156	18
18	646.752	723.719	663.489 \$	-76.967	18
17	680.457	713.146	754.616	-32.689	6
18	674.300	697.304	740.394	-23.004	4
19	636.649	674.845	670.146 \$	-38.196	8
20	588.468	654.994	670.194	-66.526	15

AAC<sub>RF</sub> = Actual average cost replacing when failure occurs

DIFFERENCE = AAC<sub>PH</sub> - AAC<sub>NONP</sub>

\* positive rank

$T^+ = 2$  ;

p-value < 0.005

\$ AAC<sub>RF</sub> is less than only AAC<sub>NONP</sub>

**Table 4 COMPARISON OF ACTUAL AVERAGE COSTS FOR SMALL SAMPLE SIZES (N = 50) OF PH DISTRIBUTION WITH  $T_{42}$  AND  $\alpha_{42}$**

Run	$AAC_{PH}$	$AAC_{NONP}$	$AAC_{RF}$	DIFFERENCE	RANK
1	618.463	561.050	545.871 !	57.413	16 *
2	604.098	684.564	686.650	-80.466	19
3	674.353	664.878	637.614 !	9.475	5 *
4	605.528	559.966	575.315	45.562	18 *
5	580.054	521.136	535.683	58.918	17 *
6	547.534	553.458	556.857	-5.924	4
7	632.508	625.711	625.711	6.797	5 *
9	632.510	647.089	651.264	-14.579	8
9	490.099	578.531	566.695 \$	-88.432	10
10	629.150	668.268	670.786	-39.118	13
11	535.032	536.771	535.194	-1.739	8
12	605.358	644.458	645.689	-39.100	11
13	519.537	552.440	552.443	-32.903	10
14	532.469	553.664	569.868	-21.195	8
15	671.880	672.507	672.160 \$	-0.627	8
16	645.125	571.650	576.024	73.475	18 *
17	495.932	533.334	539.149	-37.402	11
18	648.741	677.909	682.898	-29.168	8
19	577.580	576.101	600.152	1.479	2 *
20	560.629	519.418	532.841	41.211	14 *

$AAC_{RF}$  = Actual average cost replacing when failure occurs

DIFFERENCE =  $AAC_{PH} - AAC_{NONP}$

\* positive rank

$T^+ = 93$  ;

p-value > 0.05

\$  $AAC_{RF}$  is less than only  $AAC_{NONP}$

!  $AAC_{RF}$  is less than  $AAC_{PH}$  and  $AAC_{NONP}$

**Table 5 COMPARISON OF ACTUAL AVERAGE COSTS FOR MEDIUM SAMPLE SIZES (N = 100) OF PH DISTRIBUTION WITH  $T_{42}$  AND  $\alpha_{42}$**

Run	AAC <sub>PH</sub>	AAC <sub>NONP</sub>	AAC <sub>RF</sub>	DIFFERENCE	RANK
2	573.310	548.751	539.868 !	24.559	13 *
2	588.501	633.798	623.661 \$	-45.297	16
3	618.891	626.148	608.227 !	-7.257	2
4	621.602	590.532	601.378	31.070	14 *
5	557.189	521.040	535.590	36.149	15 *
6	549.717	562.877	569.251	-13.160	9
7	651.656	629.631	630.823	22.025	18 *
8	615.798	624.193	628.347	-8.395	4
9	475.757	574.707	567.688 \$	-98.95	20
10	602.762	626.458	628.035	-23.696	12
11	535.426	537.073	549.888	-1.647	1
12	580.480	598.790	590.995 \$	-18.310	7
13	525.770	597.208	597.210	-71.438	19
14	518.669	540.978	553.372	-22.309	11
15	634.978	627.571	633.019	7.407	3 *
16	675.027	616.239	618.041	58.788	18 *
17	535.911	585.289	587.998	-49.378	17
18	617.194	636.912	645.975	-19.718	9
19	550.468	564.214	579.831	-13.746	6
20	539.256	520.034	531.189	19.222	8 *

AAC<sub>RF</sub> = Actual average cost replacing when failure occurs

DIFFERENCE = AAC<sub>PH</sub> - AAC<sub>NONP</sub>

\* positive rank

$T^+ = 81$  ;

p-value > 0.05

\$ AAC<sub>RF</sub> is less than only AAC<sub>NONP</sub>

! AAC<sub>RF</sub> is less than AAC<sub>PH</sub> and AAC<sub>NONP</sub>



**Table 6 COMPARISON OF ACTUAL AVERAGE COSTS FOR LARGE SAMPLE SIZES (N = 200) OF PH DISTRIBUTION WITH  $T_{42}$  AND  $\alpha_{42}$**

Run	$AAC_{PH}$	$AAC_{NONP}$	$AAC_{RF}$	DIFFERENCE	RANK
1	556.373	550.131	544.302 !	6.242	3 *
2	574.206	616.553	621.391	-42.347	17
3	578.172	591.186	606.516	-13.014	7
3	594.573	579.915	588.547	14.658	8 *
5	580.261	551.976	560.463	28.285	14 *
6	526.351	558.604	573.294	-32.253	15
7	632.238	622.079	624.275	10.159	8 *
8	612.338	621.511	610.932 !	-9.173	5
9	460.437	567.069	563.305 \$	-106.632	20
10	557.424	599.181	602.122	-41.757	10
11	555.711	539.303	551.773	16.408	8 *
12	560.534	584.128	589.078	-23.594	12
13	576.939	621.856	621.857	-44.917	18
14	568.925	595.619	599.305	-26.694	19
15	562.097	583.057	595.668	-20.960	11
16	584.778	604.211	606.979	-19.433	10
17	549.979	607.061	610.878	-57.082	19
16	631.719	628.295	639.159	3.424	8 *
19	578.172	577.936	607.344	0.236	1 *
20	538.683	530.002	570.516	8.681	4 *

$AAC_{RF}$  = Actual average cost replacing when failure occurs

DIFFERENCE =  $AAC_{PH} - AAC_{NONP}$

\* positive rank

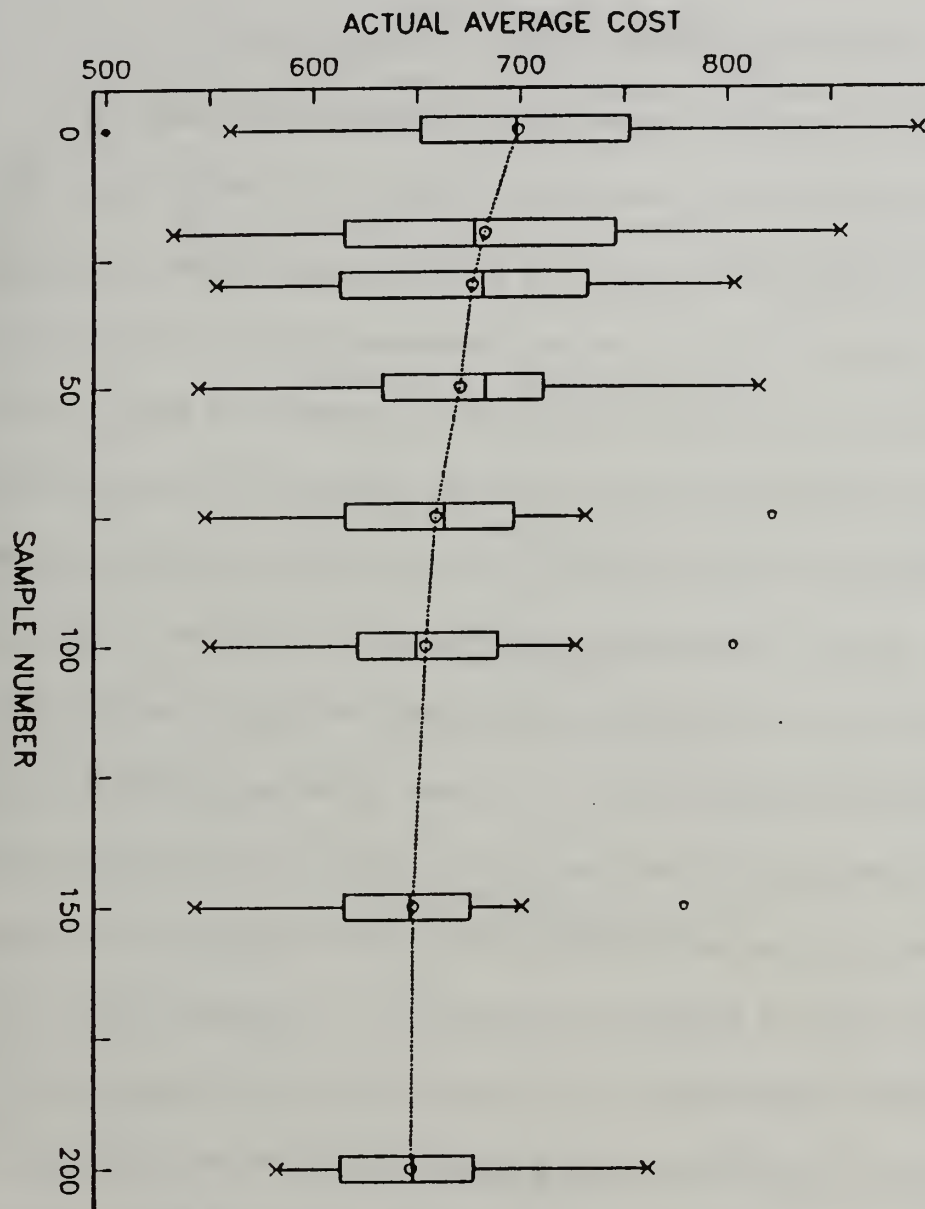
$T^+ = 47$  ;

$0.01 < p\text{-value} < 0.025$

\$  $AAC_{RF}$  is less than only  $AAC_{NONP}$

!  $AAC_{RF}$  is less than  $AAC_{PH}$  and  $AAC_{NONP}$

# PHASE TYPE SEQUENTIAL ESTIMATION PROCEDURE



**Figure 5** The box plot of actual average cost for different sample sizes of phase type sequential procedure for  $T_{41}$  and  $\alpha_{41}$

# NONPARAMETRIC SEQUENTIAL ESTIMATION PROCEDURE

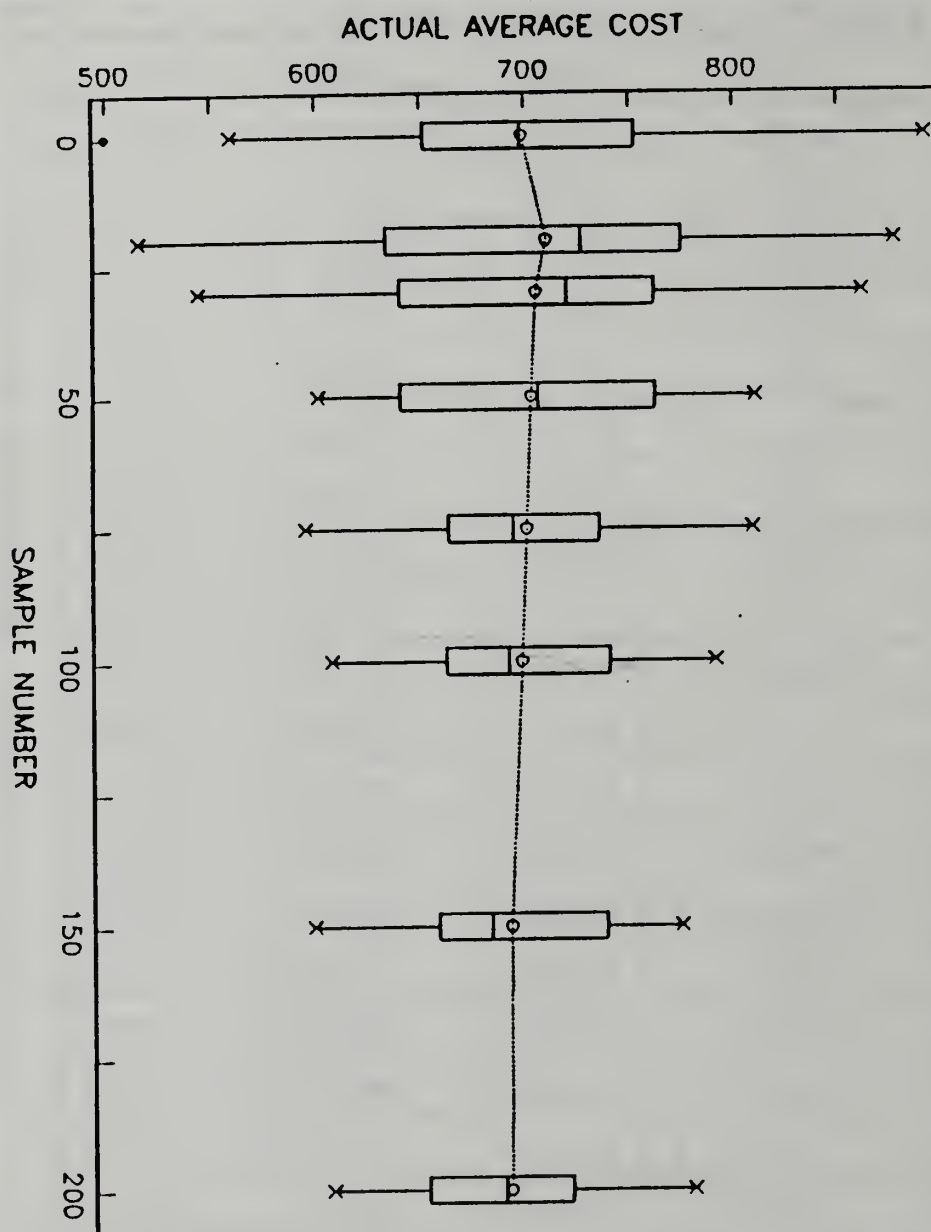


Figure 6 The box plot of actual average cost for different sample sizes of nonparametric sequential procedure for  $T_{41}$  and  $\alpha_{41}$



## V.CONCLUSIONS AND RECOMMENDATIONS

In this thesis, the age replacement policy has been considered. A system is replaced at the time of failure or at a scheduled replacement age whichever comes first. The replacement system is assumed to be as good as new. The objective is to achieve the minimum long run expected maintenance cost per unit time. The system lifetimes are assumed to have PH distributions. Estimating the parameters of a PH distribution is difficult because it may have many parameters and its density function involves a matrix exponential. Moment matching with a nonlinear programming approach is used for estimating the parameters. This method does not guarantee that the estimated cost function will have a unique and finite minimum. In this thesis we restrict our attention to the case with initial probability vector  $\alpha = (1, 0, 0)$ . Sequential estimation using the phase type procedure gives smaller average costs than the nonparametric procedure for both small and large sample sizes when the underlying long run average cost function has a very distinct minimum at  $\phi^*$ . When this cost function is shallow around  $\phi^*$ , the parametric procedure does not do better than the nonparametric procedure for small samples. In fact, for cost functions that are shallow, it is better not to use a maintenance policy i.e. to take  $\hat{\phi}^* = \infty$ , until the sample sizes are large enough to give reasonable estimates of the underlying parameters. The reason for this is that for such cost functions, overestimating  $\phi^*$  does not increase the long run average cost much, at the same time the decrease in the amount of censoring with over estimating  $\phi^*$  greatly

improves estimation of the underlying distribution function  $F$ . Conversely underestimating  $\phi^*$  for these cost functions causes a drastic increase in long run average costs and increase censoring making estimation very difficult.

### **Recommendation for the future research**

The following forms a list of future research initiatives:

- Examine the impact of the product-limit estimator in estimating standardized moments on the rest of procedure.

- Do more experimentation on different representations of phase type distributions to get more precise conclusion.

- Look for other policies using phase-sensitive estimate and modified replacement costs over time.

- The phase-type sequential estimations we have been so far assume the underlying lifetimes are iid and after replacement the system is new. To be more realistic, we can modify the initial probability vector, increase transition rates between states, or both, over time.

- Use phase-type sequential estimation when the underlying life distribution  $F$  comes from Gamma, Weibull etc., that have increasing failure rate and compare with the nonparametric procedure. Since we can use the denseness of phase type distribution property to approximate the set of distribution with support on  $[0, \alpha)$  i.e., the set of lifetimes, we may get a better method when the distribution of lifetimes is not known.

## APPENDIX A.

\$TITLE PH DISTRIBUTION ESTIMATION PARAMETERS

\*

\*-----GAMS AND DOLLAR CONTROL OPTIONS-----

\*(See Appendices B & C)

### OPTIONS

WORK = 100000,

LIMCOL = 0 , LIMROW = 0 , SOLPRINT = OFF , DECIMALS = 2

RESLIM = 100, ITERLIM = 100000, OPTCR = 0.0 , SEED = 3141;

\*-----DEFINITIONS AND DATA-----

\* This program uses nonlinear programming to estimate the 3-transient states of phase

\* type distribution parameters. Matching the second and third standardized moments are

\* used. All adjustable data are prepared by the "FILE SCALAR2" and uses the

\* "\$INCLUDE" statement bring to the program.

\$INCLUDE "FILE SCALAR2"

\*-----MODEL-----

### VARIABLE

Z objective function value

A first entry of the matrix

B second entry of the matrix

C third entry of the matrix

D fourth entry of the matrix

E fifth entry of the matrix

F sixth entry of the matrix

G seventh entry of the matrix

H eighth entry of the matrix

I ninth entry of the matrix

AL1 initial probability of being in state 1

AL2 initial probability of being in state 2

AL3 initial probability of being in state 3 ;

### EQUATION

OBJ Define objective function

EQ1 Set initial probability of being in state 1(nonnegative)

EQ2 Set initial probability of being in state 2(nonnegative)

EQ3 Set initial probability of being in state 3(nonnegative)  
 EQ4 Sum over all of initial prob. less than or equal 1.0  
 EQ5 Set the first diagonal entry to be nonpositive  
 EQ6 Set the second diagonal entry to be nonpositive  
 EQ7 Set the third diagonal entry to be nonpositive  
 EQ8 Set off diagonal entries in the first row to be positive  
 EQ9 Set off diagonal entries in the first row to be positive  
 EQ10 Set sum off diagonal of the first row less than or equal first diagonal  
 EQ11 Set off diagonal entries in the second row to be positive  
 EQ12 Set off diagonal entries in the second row to be positive  
 EQ13 Set sum off diagonal of the second row less than or equal second diagonal  
 EQ14 Set off diagonal entries in the third row to be positive  
 EQ15 Set off diagonal entries in the third row to be positive  
 EQ16 Set sum off diagonal of the third row less than or equal third diagonal  
 EQ17 Define the expected lifetime ;

\* MINIMIZE

OBJ..

Z =E=

\*\*\* 3\*SQR(C - C(0)) \*\*\*

3\*POWER((((2\*AL1\*(POWER((-F\*I)+E\*I),2)+(C\*I-B\*I)\*(F\*I-D\*I)  
 +(-(C\*I)+B\*I)\*(-(E\*I)+D\*I)  
 +(-(F\*I)+E\*I)\*(C\*I-B\*I)+(C\*I-B\*I)\*(-(C\*I)+A\*I)+(-(C\*I)+B\*I)\*(B\*I-A\*I)  
 +(-(F\*I)+E\*I)\*(-(C\*I)+B\*I)+(C\*I-B\*I)\*(C\*I-A\*I)+(-(C\*I)+B\*I)\*(-(B\*I)+A\*I))  
 + 2\*AL2\*((F\*I-D\*I)\*(-(F\*I)+E\*I)+(-(C\*I)+A\*I)\*(F\*I-D\*I)+(-(E\*I)+D\*I)\*  
 (C\*I-A\*I)  
 +(F\*I-D\*I)\*(C\*I-B\*I)+POWER((-C\*I)+A\*I,2)+(C\*I-A\*I)\*(B\*I-A\*I)  
 +(F\*I-D\*I)\*(-(C\*I)+B\*I)+(-(C\*I)+A\*I)\*(C\*I-A\*I)+(C\*I-A\*I)\*(-(B\*I)+A\*I))  
 + 2\*AL3\*((-(E\*I)+D\*I)\*(-(F\*I)+E\*I)+(B\*I-A\*I)\*(F\*I-D\*I)+(-(B\*I)+A\*I)\*  
 (-(E\*I)+D\*I)  
 +(-(E\*I)+D\*I)\*(C\*I-B\*I)+(B\*I-A\*I)\*(-(C\*I)+A\*I)+(-(B\*I)+A\*I)\*  
 (B\*I-A\*I)+(-(E\*I)+D\*I)\*(-(C\*I)+B\*I)+(B\*I-A\*I)\*(C\*I-A\*I)+  
 POWER((-B\*I)+A\*I,2))))/  
 POWER((-C\*I)+B\*I+ C\*I-D\*I - A\*I-F\*I - B\*I-D\*I + A\*I-E\*I),2)  
 -POWER((-AL1\*(-F\*I+E\*I+C\*I-B\*I-(C\*I)+B\*I)  
 +AL2\*(F\*I-D\*I-C\*I+ A\*I+C\*I-A\*I)+AL3\*(-E\*I+D\*I+B\*I-A\*I-B\*I-D\*I+A\*I))/



$$(-(C^*E^*G)+B^*F^*G+C^*D^*H-A^*F^*H-B^*D^*I+A^*E^*I),2))^{**0.5} /$$

$$\begin{aligned} &(- (AL1^*(-(F^*H)+E^*I+C^*H-B^*I-(C^*E)+B^*F) \\ &+AL2^*(F^*G-D^*I-C^*G+A^*I+C^*D-A^*F) \\ &+AL3^*(-E^*G+D^*H+B^*G-A^*H-B^*D+A^*E)))/ \\ &(-(C^*E^*G) + B^*F^*G + C^*D^*H - A^*F^*H - B^*D^*I + A^*E^*I)) - MM2),2) \end{aligned}$$

$$*** W2*SQR(GAMMA - GAMMA(0)) ***$$

$$+ POWER(((((-6*AL1*((POWER((- (F^*H)+E^*I),2)+(C^*H-B^*I)*(F^*G-D^*I) + (- (C^*E)+B^*F)*(-(E^*G)+D^*H))*(-(F^*H)+E^*I)$$

$$+(((-(F^*H)+E^*I)*(C^*H-B^*I)+(C^*H-B^*I)*(-(C^*G)+A^*I)+(-(C^*E)+B^*F)* (B^*G-A^*H))* (F^*G-D^*I)$$

$$+(((-(F^*H)+E^*I)*(-(C^*E)+B^*F)+(C^*H-B^*I)*(C^*D-A^*F)+(-(C^*E)+B^*F)* (- (B^*D)+A^*E))*(-(E^*G)+D^*H)$$

$$+(POWER((- (F^*H)+E^*I),2)+(C^*H-B^*I)*(F^*G-D^*I)+(-(C^*E)+B^*F)* (- (E^*G)+D^*H))* (C^*H-B^*I)$$

$$+(((-(F^*H)+E^*I)*(C^*H-B^*I)+(C^*H-B^*I)*(-(C^*G)+A^*I)+(-(C^*E)+B^*F)*(B^*G-A^*H))* (- (C^*G)+A^*I)$$

$$+(((-(F^*H)+E^*I)*(-(C^*E)+B^*F)+(C^*H-B^*I)*(C^*D-A^*F)+(-(C^*E)+B^*F)* (- (B^*D)+A^*E))* (B^*G-A^*H)$$

$$+(POWER((- (F^*H)+E^*I),2)+(C^*H-B^*I)*(F^*G-D^*I)+(-(C^*E)+B^*F)* (- (E^*G)+D^*H))*(-(C^*E)+B^*F)$$

$$+(((-(F^*H)+E^*I)*(C^*H-B^*I)+(C^*H-B^*I)*(-(C^*G)+A^*I)+(-(C^*E)+B^*F)* (B^*G-A^*H))* (C^*D-A^*F)$$

$$+(((-(F^*H)+E^*I)*(-(C^*E)+B^*F)+(C^*H-B^*I)*(C^*D-A^*F)+(-(C^*E)+B^*F)* (- (B^*D)+A^*E))*(-(B^*D)+A^*E))$$

$$-6*AL2^*((F^*G-D^*I)*(-(F^*H)+E^*I)+(-(C^*G)+A^*I)*(F^*G-D^*I)+(-(E^*G)+D^*H)* (C^*D-A^*F))*(-(F^*H)+E^*I)$$

$$+((F^*G-D^*I)*(C^*H-B^*I)+POWER((- (C^*G)+A^*I),2)+(C^*D-A^*F)*(B^*G-A^*H))* (F^*G-D^*I)$$

$$+((F^*G-D^*I)*(-(C^*E)+B^*F)+(-(C^*G)+A^*I)*(C^*D-A^*F)+(C^*D-A^*F)*$$

$$\begin{aligned}
& (- (B * D) + A * E) * (- (E * G) + D * H) \\
& + ((F * G - D * I) * (- (F * H) + E * I) + (- (C * G) + A * I) * (F * G - D * I) + (- (E * G) + D * H) * \\
& \quad (C * D - A * F)) * (C * H - B * I) \\
& + ((F * G - D * I) * (C * H - B * I) + \text{POWER}((- (C * G) + A * I), 2) + (C * D - A * F) * (B * G - A * H)) * \\
& \quad (- (C * G) + A * I) \\
& + ((F * G - D * I) * (- (C * E) + B * F) + (- (C * G) + A * I) * (C * D - A * F) + (C * D - A * F) * \\
& \quad (- (B * D) + A * E)) * (B * G - A * H) \\
& + ((F * G - D * I) * (- (F * H) + E * I) + (- (C * G) + A * I) * (F * G - D * I) + (- (E * G) + D * H) * \\
& \quad (C * D - A * F)) * (- (C * E) + B * F) \\
& + ((F * G - D * I) * (C * H - B * I) + \text{POWER}((- (C * G) + A * I), 2) + (C * D - A * F) * (B * G - A * H)) * \\
& \quad (C * D - A * F) \\
& + ((F * G - D * I) * (- (C * E) + B * F) + (- (C * G) + A * I) * (C * D - A * F) + (C * D - A * F) * \\
& \quad (- (B * D) + A * E)) * (- (B * D) + A * E) \\
& - 6 * \text{AL3} * (((- (E * G) + D * H) * (- (F * H) + E * I) + (B * G - A * H) * (F * G - D * I) + (- (B * D) + A * E) * \\
& \quad (- (E * G) + D * H)) * (- (F * H) + E * I) \\
& + (((- (E * G) + D * H) * (C * H - B * I) + (B * G - A * H) * (- (C * G) + A * I) + (- (B * D) + A * E) * \\
& \quad (B * G - A * H)) * (F * G - D * I) \\
& + (((- (E * G) + D * H) * (- (C * E) + B * F) + (B * G - A * H) * (C * D - A * F) + \\
& \quad \text{POWER}((- (B * D) + A * E), 2)) * (- (E * G) + D * H) \\
& + (((- (E * G) + D * H) * (- (F * H) + E * I) + (B * G - A * H) * (F * G - D * I) + (- (B * D) + A * E) * \\
& \quad (- (E * G) + D * H)) * (C * H - B * I) \\
& + (((- (E * G) + D * H) * (C * H - B * I) + (B * G - A * H) * (- (C * G) + A * I) + (- (B * D) + A * E) * \\
& \quad (B * G - A * H)) * (- (C * G) + A * I) \\
& + (((- (E * G) + D * H) * (- (C * E) + B * F) + (B * G - A * H) * (C * D - A * F) + \\
& \quad \text{POWER}((- (B * D) + A * E), 2)) * (B * G - A * H) \\
& + (((- (E * G) + D * H) * (- (F * H) + E * I) + (B * G - A * H) * (F * G - D * I) + (- (B * D) + A * E) * \\
& \quad (- (E * G) + D * H)) * (- (C * E) + B * F) \\
& + (((- (E * G) + D * H) * (C * H - B * I) + (B * G - A * H) * (- (C * G) + A * I) + (- (B * D) + A * E) * \\
& \quad (B * G - A * H)) * (C * D - A * F)
\end{aligned}$$



$$+((-E*G)+D*H)*(-(C*E)+B*F)+(B*G-A*H)*(C*D-A*F)+$$

$$POWER((-B*D)+A*E,2))*(-(B*D)+A*E)))$$

$$/POWER((-C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),3)$$

$$+3*((AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)+AL2*(F*G-D*I-C*G+A*I+C*D-A*F)+AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/(-C*E*G+B*F*G+C*D*H-A*F*H-B*D*I+A*E*I))*$$

\*\*\* expected of lifetime square \*\*\*

$$((2*AL1*(POWER((-F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)+(-(C*E)+B*F)*$$

$$(-(E*G)+D*H)+(-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+$$

$$(-(C*E)+B*F)*(B*G-A*H)+(-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)$$

$$+(-(C*E)+B*F)*(-(B*D)+A*E))$$

$$+2*AL2*((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)$$

$$+(-(E*G)+D*H)*(C*D-A*F)+(F*G-D*I)*(C*H-B*I)+POWER((-C*G)+A*I),2)$$

$$+(C*D-A*F)*(B*G-A*H)+(F*G-D*I)*(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)$$

$$+(C*D-A*F)*(-(B*D)+A*E))$$

$$+2*AL3*((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)$$

$$+(-(B*D)+A*E)*(-(E*G)+D*H)+(-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*$$

$$(-(C*G)+A*I)+(-(B*D)+A*E)*(B*G-A*H)+(-(E*G)+D*H)*(-(C*E)+B*F)$$

$$+(B*G-A*H)*(C*D-A*F)+POWER((-B*D)+A*E,2)))/$$

$$POWER((-C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),2))$$

\*\*\* 2 time cube of the expected lifetime \*\*\*

$$-2*POWER(((AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)$$

$$+AL2*(F*G-D*I-C*G+A*I+C*D-A*F)+AL3*(-E*G+D*H+B*G-A*H-B*D+A*E))/$$

$$(-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I)),3))$$

$$/((2*AL1*(POWER((-F*H)+E*I),2)+(C*H-B*I)*(F*G-D*I)+(-(C*E)+B*F)*$$

$$(-(E*G)+D*H)+(-(F*H)+E*I)*(C*H-B*I)+(C*H-B*I)*(-(C*G)+A*I)+$$

$$(-(C*E)+B*F)*(B*G-A*H)+(-(F*H)+E*I)*(-(C*E)+B*F)+(C*H-B*I)*(C*D-A*F)$$

$$+(-(C*E)+B*F)*(-(B*D)+A*E))$$

$$+2*AL2*((F*G-D*I)*(-(F*H)+E*I)+(-(C*G)+A*I)*(F*G-D*I)+$$

$$(-(E*G)+D*H)*(C*D-A*F)+(F*G-D*I)*(C*H-B*I)+$$

$$POWER((-C*G)+A*I),2)+(C*D-A*F)*(B*G-A*H)+(F*G-D*I)*$$

$$(-(C*E)+B*F)+(-(C*G)+A*I)*(C*D-A*F)+(C*D-A*F)*(-(B*D)+A*E))$$

$$+ 2*AL3*((-(E*G)+D*H)*(-(F*H)+E*I)+(B*G-A*H)*(F*G-D*I)+$$

$$(-(B*D)+A*E)*(-(E*G)+D*H)+(-(E*G)+D*H)*(C*H-B*I)+(B*G-A*H)*$$

$$(-(C*G)+A*I)+(-(B*D)+A*E)*(B*G-A*H)+(-(E*G)+D*H)*(-(C*E)+B*F)$$

$$+(B*G-A*H)*(C*D-A*F)+POWER((-(B*D)+A*E),2))/$$

$$(POWER((-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),2))$$

$$-POWER((((AL1*(-(F*H)+E*I+C*H-B*I-(C*E)+B*F)$$

$$+AL2*(F*G-D*I-C*G+A*I+C*D-A*F)$$

$$+AL3*(-(E*G)+D*H+B*G-A*H-B*D+A*E))/$$

$$(-(C*E*G) + B*F*G + C*D*H - A*F*H - B*D*I + A*E*I),2))**1.5) - MM3),2);$$

\*

\* SUBJECT TO

\*

EQ1..

$$AL1 = G = 0 ;$$

EQ2..

$$AL2 = G = 0 ;$$

EQ3..

$$AL3 = G = 0 ;$$

EQ4..

$$AL1 + AL2 + AL3 = L = 1 ;$$

EQ5..

$$A = L = 0 ;$$

EQ6..

$$E = L = 0 ;$$

EQ7..

$$I = L = 0 ;$$

EQ8..

$$B = G = 0 ;$$

EQ9..

$$C = G = 0 ;$$

EQ10..

$$B + C = L = -A ;$$

EQ11..

$$D = G = 0 ;$$

EQ12..

$$F = G = 0 ;$$

EQ13..

$$D + F = L = -E ;$$

EQ14..

$$G = G = 0 ;$$

EQ15..

H =G= 0 ;

EQ16..

G + H =L= -I ;

EQ17..

$$\frac{-(AL1 * (-(F * H) + E * I + C * H - B * I - (C * E) + B * F) + AL2 * (F * G - D * I - C * G + A * I + C * D - A * F) + AL3 * (-E * G + D * H + B * G - A * H - B * D + A * E))}{(-(C * E * G) + B * F * G + C * D * H - A * F * H - B * D * I + A * E * I)} = E = EX ;$$

\*

\*-----REPORT-----

\*

MODEL MATCHM /ALL/ ;

\* Parameters bounding

A.LO = -7.0; B.LO = 3.0; C.LO = 0.0; D.LO = 0.0; E.LO = -7.0;  
A.UP = -5.0; B.UP = 5.0; C.UP = 2.0; D.UP = 1.0; E.UP = -5.0;  
F.LO = 4.0; G.LO = 0.0; H.LO = 0.0; I.LO = -8.0;  
F.UP = 5.0; G.UP = 1.0; H.UP = 2.0; I.UP = -7.0;  
AL1.LO = 1.0; AL2.LO = 0.0; AL3.LO = 0.0;  
AL1.UP = 1.0; AL2.UP = 0.0; AL3.UP = 0.0;

\* The initial solution

A.L = AA ; B.L = BB ; C.L = CC; D.L = DD; E.L = EE ;  
F.L = FF ; G.L = GG ; H.L = HH ; I.L = II ;  
AL1.L = ALP1 ; AL2.L = ALP2 ; AL3.L = ALP3 ;

SOLVE MATCHM USING NLP MINIMIZING Z ;  
DISPLAY Z.L, A.L , B.L, C.L, D.L, E.L, F.L, G.L, H.L, I.L,  
AL1.L, AL2.L, AL3.L;

\* Write output to the CMS file

FILE OUT1 /FILE OBJ A / ;  
PUT OUT1 ;  
PUT Z.L ;  
FILE OUT2 /FILE ALPHA A / ;  
PUT OUT2 ;  
PUT A.L ;  
FILE OUT3 /FILE BRAVO A / ;  
PUT OUT3 ;

PUT B.L ;  
FILE OUT4 /FILE CHAREE A / ;  
PUT OUT4 ;  
PUT C.L ;  
FILE OUT5 /FILE DELTA A / ;  
PUT OUT5 ;  
PUT D.L ;  
FILE OUT6 /FILE ECHO A / ;  
PUT OUT6 ;  
PUT E.L ;  
FILE OUT7 /FILE FOXTROT A / ;  
PUT OUT7 ;  
PUT F.L ;  
FILE OUT8 /FILE GOLF A / ;  
PUT OUT8 ;  
PUT G.L ;  
FILE OUT9 /FILE HOTEL A / ;  
PUT OUT9 ;  
PUT H.L ;  
FILE OUT10 /FILE INDIA A / ;  
PUT OUT10 ;  
PUT I.L ;  
FILE OUT11 /FILE AL1 A / ;  
PUT OUT11 ;  
PUT AL1.L ;  
FILE OUT12 /FILE AL2 A / ;  
PUT OUT12 ;  
PUT AL2.L ;  
FILE OUT13 /FILE AL3 A / ;  
PUT OUT13 ;  
PUT AL3.L ;

# THE INITIAL COMPLETE SYSTEM LIFETIME SET 1-5

N	SET 1	SET 2	SET 3	SET 4	SET 5
1	0.1226	0.0740	0.6855	0.0528	0.1143
2	0.1455	0.1413	0.2157	0.1139	0.1999
3	0.2662	0.2261	0.2209	0.2412	0.2002
4	0.2925	0.2629	0.2839	0.2892	0.2422
5	0.3205	0.2831	0.2852	0.3118	0.3164
6	0.4214	0.3340	0.3116	0.3485	0.3398
7	0.4394	0.3911	0.3403	0.3973	0.3534
8	0.4685	0.4517	0.4517	0.4442	0.3893
9	0.5804	0.4761	0.4691	0.4686	0.4410
10	0.5871	0.4909	0.5254	0.4777	0.4766
11	0.6228	0.5290	0.5536	0.4885	0.4964
12	0.6534	0.5818	0.5993	0.4970	0.5074
13	0.7220	0.5962	0.6575	0.5114	0.5240
14	0.7590	0.6233	0.6623	0.6073	0.5825
15	0.7618	0.6419	0.6855	0.6433	0.6033
16	0.8206	0.7407	0.7156	0.7576	0.6255
17	0.8245	0.7682	0.7690	0.8057	0.7391
18	0.8317	0.8312	0.7866	0.8740	0.7723
19	1.0056	0.8693	0.2852	1.1004	0.9128
20	1.0853	0.9727	0.9473	1.0424	1.0832
21	1.0853	1.1084	1.0478	1.0634	1.1936
22	1.1659	1.1272	1.1155	1.0659	1.2003
23	1.1852	1.2037	1.2936	1.2238	1.8976
24	1.2040	1.3319	1.3906	1.0424	2.0769
25	1.3069	2.5362	1.6715	3.0033	2.2631
E[X]	0.705	0.704	0.679	0.735	0.742
2 <sup>nd</sup> MM	0.47917	0.70936	0.57558	0.80566	0.77290
3 <sup>rd</sup> MM	0.03190	1.88276	0.73612	2.23985	1.40912



# THE INITIAL COMPLETE SYSTEM LIFETIME SET 6-10

N	SET 6	SET 7	SET 8	SET 9	SET 10
1	0.1337	0.0402	0.1759	0.0830	0.0761
2	0.1538	0.0634	0.2031	0.1101	0.0913
3	0.2452	0.1939	0.2106	0.2083	0.2235
4	0.2876	0.1954	0.2376	0.2339	0.3309
9	0.3591	0.1958	0.3043	0.2408	0.3470
8	0.4589	0.2353	0.3719	0.2726	0.3662
7	0.5161	0.2775	0.3850	0.2777	0.4091
8	0.6030	0.3185	0.3892	0.2894	0.4523
9	0.6181	0.3579	0.4032	0.3137	0.5581
13	0.6761	0.3922	0.4199	0.3227	0.5800
18	0.6851	0.3985	0.6029	0.3390	0.5805
12	0.7738	0.5206	0.6406	0.3878	0.5993
13	0.8541	0.5233	0.6506	0.3878	0.6354
19	0.8654	0.6261	0.7156	0.4055	0.6571
18	0.4589	0.6336	0.7584	0.4076	0.7529
18	1.0193	0.6776	0.7677	0.4333	0.7874
17	1.0352	0.8705	0.8754	0.6178	0.7991
18	1.0677	1.1124	0.8830	0.6245	0.8572
19	1.1196	1.1228	0.9575	0.6455	0.8920
20	1.1995	1.1742	0.9805	0.9824	0.9024
21	1.2301	1.1872	1.3367	1.0054	0.9699
22	1.3043	1.1887	1.4040	1.1143	0.9734
23	2.0273	1.4849	1.5844	1.2279	1.0452
20	3.2891	1.7218	1.6250	1.2420	1.1461
25	3.5360	2.5301	1.8392	2.1319	1.1594
E[X]	1.000	0.722	0.749	0.572	0.648
2 <sup>nd</sup> MM	0.82457	0.81263	0.63261	0.80948	0.46844
3 <sup>rd</sup> MM	1.89734	1.25687	0.79102	1.69217	0.14707



# THE INITIAL COMPLETE SYSTEM LIFETIME SET 11-15

N	SET 11	SET 12	SET 13	SET 14	SET 15
1	0.0268	0.9433	0.0470	0.1730	0.1101
2	0.0745	0.1285	0.0922	0.2556	0.1386
9	0.1235	0.2052	0.1295	0.3124	0.1709
1	0.1800	0.2435	0.2064	0.3125	0.1998
8	0.2526	0.2924	0.2924	0.3447	0.2075
8	0.2704	0.3335	0.2957	0.3894	0.2588
8	0.2972	0.3645	0.3110	0.4044	0.2669
8	0.3697	0.4936	0.3209	0.5559	0.2737
9	0.3811	0.4984	0.4173	0.5796	0.2508
10	0.3990	0.5064	0.4577	0.5803	0.3506
11	0.4070	0.5778	0.5168	0.5928	0.3604
12	0.4855	0.6699	0.5179	0.6056	0.4117
13	0.6402	0.7442	0.5258	1.6525	0.5809
10	0.7087	0.7502	0.6056	0.7299	0.5981
15	0.7408	0.8656	0.5547	0.7355	0.6558
10	0.8496	0.8831	0.5559	0.8122	0.7306
10	0.9290	1.0607	0.5633	0.9379	0.7388
12	0.9873	1.1003	0.5692	0.9433	0.7400
15	0.9983	1.1834	0.6501	1.0197	0.7786
20	1.0318	1.2235	0.7978	1.2675	0.7936
21	1.0937	1.3370	0.8750	1.3842	1.1532
22	1.1137	1.5754	1.0843	1.6525	1.2444
23	1.2706	1.8085	1.1683	1.8276	1.2781
24	1.4875	1.9023	1.1881	1.8545	1.3428
25	1.4958	1.9894	1.3842	2.8764	1.5937
E[X]	0.665	0.831	0.560	0.872	0.611
2 <sup>nd</sup> MM	0.64986	0.66778	0.60540	0.71398	0.68553
3 <sup>rd</sup> MM	0.32858	0.60712	0.69085	1.51250	0.77568

# THE INITIAL COMPLETE SYSTEM LIFETIME SET 16-20

N	SET 16	SET 17	SET 18	SET 19	SET 20
1	0.1677	0.0565	0.0398	0.1798	0.0202
2	0.1819	0.1811	0.1402	0.2037	0.1920
3	0.2139	0.1941	0.1506	0.2197	0.2205
3	0.3436	0.2810	0.1798	0.3159	0.2209
5	0.3641	0.3315	0.1922	0.3464	0.3406
6	0.4393	0.4147	0.3787	0.3629	0.3645
9	0.4800	0.4249	0.6600	0.3744	0.3941
9	0.5328	0.4629	0.4944	0.3793	0.4130
9	0.5643	0.9939	0.5952	0.3870	0.5525
11	0.7220	0.5382	0.6198	0.3875	0.5729
11	0.7393	0.5776	0.6269	0.1011	0.5787
12	0.7666	0.6003	0.6276	0.4906	0.5835
19	0.7766	0.6055	0.6608	0.7436	0.6138
14	0.7964	0.6230	0.7183	0.7690	0.6389
15	0.7981	0.6477	0.7970	0.7824	0.6609
14	0.7220	0.7631	0.7997	0.8226	0.6624
17	0.8390	0.7970	0.8065	0.8492	0.7091
11	0.9422	0.7988	0.8065	0.9542	0.7178
19	1.0115	0.8518	0.9682	1.2333	0.7447
20	1.0381	0.9091	0.9939	1.3601	1.2027
21	1.1195	1.0447	1.0028	1.4489	1.3247
22	1.2716	1.1973	1.0875	1.5989	1.4412
23	1.3426	1.2331	1.2722	2.2702	1.6155
24	1.6082	1.2339	1.4913	2.2971	1.7162
25	1.7724	1.3386	1.8663	3.1519	2.1912
E[X]	0.786	0.664	0.711	0.893	0.748
2 <sup>nd</sup> MM	0.52328	0.51958	0.60066	0.83695	0.70314
3 <sup>rd</sup> MM	0.55452	0.32242	0.66254	1.46868	1.13073

## APPENDIX B.

### PROGRAM COST

```
C
C THIS PROGRAM PROVIDES THE OPTIMAL AGE REPLACEMENT OF THE
C PHASE TYPE DISTRIBUTION. IT WORKS ONLY SPECIFIC CASE OF 3
C PHASES OF TRANSIENT STATES. THE PARAMETERS ARE AS FOLLOWS:
C   T(I,J) = ENTRIES OF THE TRANSITION MATRIX OF PH DISTN.
C   AL(I)  = INITIAL PROBABILITY OF BEING IN STATE I
C   C1     = PLANNED MAINTENANCE COST
C   C2     = UNPLANNED MAINTENANCE COST
C   L      = MAXIMUM LIFETIME THAT WANT TO CALCULATE
C   DET    = DETERMINANT OF TRANSITION MATRIX
C   SMALL  = THE OPTIMAL AGE REPLACEMENT
C   CS     = AGE REPLACEMENT COSTS
```

```
*
* INITIAL INPUT
*
```

```
INTEGER I,J,K,N
REAL C1, C2, TI(3,3), T(3,3), DET, AL(3), L, OBJ
REAL TS(3,3), SMALL, PI, A, IDEN(3,3), S, NUMER
REAL DENOM, P(3,3), Q(3,3), CS(400), EXPTS(3,3)
DATA C1,C2,L,TS/100.0,500.0,4.0,9*0.0 /
N = 0
S = 0.01
OPEN(UNIT =11,FILE = 'ALPHA')
OPEN(UNIT =12,FILE = 'BRAVO')
OPEN(UNIT =13,FILE = 'CHAREE')
OPEN(UNIT =14,FILE = 'DELTA')
OPEN(UNIT =15,FILE = 'ECHO')
OPEN(UNIT =16,FILE = 'FOXTROT')
OPEN(UNIT =17,FILE = 'GOLF')
OPEN(UNIT =18,FILE = 'HOTEL')
OPEN(UNIT =19,FILE = 'INDIA')
OPEN(UNIT =20,FILE = 'AL1')
OPEN(UNIT =21,FILE = 'AL2')
OPEN(UNIT =22,FILE = 'AL3')
OPEN(UNIT =23,FILE = 'OBJ')
READ(11,*)T(1,1)
READ(12,*)T(1,2)
READ(13,*)T(1,3)
```

```

READ(14,*)T(2,1)
READ(15,*)T(2,2)
READ(16,*)T(2,3)
READ(17,*)T(3,1)
READ(18,*)T(3,2)
READ(19,*)T(3,3)
READ(20,*)AL(1)
READ(21,*)AL(2)
READ(22,*)AL(3)
READ(23,*)OBJ
IF(OBJ .GT. 0.05) GOTO 555

```

\*

\* CALCULATION COST FOR EACH REPLACEMENT TIME

\*

```

DET = -(T(1,3)*T(2,2)*T(3,1))
/      + T(1,2)*T(2,3)*T(3,1)
/      + T(1,3)*T(2,1)*T(3,2)
/      - T(1,1)*T(2,3)*T(3,2)
/      - T(1,2)*T(2,1)*T(3,3)
/      + T(1,1)*T(2,2)*T(3,3)
PRINT*, 'DET = ', DET
TI(1,1) = (-T(2,3)*T(3,2) + T(2,2)*T(3,3))/DET
TI(1,2) = (T(1,3)*T(3,2) - T(1,2)*T(3,3))/DET
TI(1,3) = (-T(1,3)*T(2,2) + T(1,2)*T(2,3))/DET
TI(2,1) = (T(2,3)*T(3,1) - T(2,1)*T(3,3))/DET
TI(2,2) = (-T(1,3)*T(3,1) + T(1,1)*T(3,3))/DET
TI(2,3) = (T(1,3)*T(2,1) - T(1,1)*T(2,3))/DET
TI(3,1) = (-T(2,2)*T(3,1) + T(2,1)*T(3,2))/DET
TI(3,2) = (T(1,2)*T(3,1) - T(1,1)*T(3,2))/DET
TI(3,3) = (-T(1,2)*T(2,1) + T(1,1)*T(2,2))/DET
DO 10 I = 1,3
    DO 5 J = 1,3
        IF(I.EQ.J)THEN
            IDEN(I,J) = 1.0
        ELSE
            IDEN(I,J) = 0.0
        ENDIF
    5 CONTINUE
10 CONTINUE

```

\*

100 CONTINUE



```

      N = N+1
      DO 20 I = 1,3
        DO 15 J = 1,3
          TS(I,J) = S*T(I,J)
15    CONTINUE
20 CONTINUE
*
* FIND THE EXPONENTIAL OF MATRIX TS
*
      CALL EXPON(TS,EXPTS,PI)
      A = 0.0
      DO 21 I = 1,3
        DO 22 J = 1,3
          A = A + AL(I)*EXPTS(I,J)
22    CONTINUE
21 CONTINUE
      NUMER = C2 - A*(C2-C1)
      DO 30 I = 1,3
        DO 25 J = 1,3
          P(I,J) = EXPTS(I,J) - IDEN(I,J)
25    CONTINUE
30 CONTINUE
      DO 45 I = 1,3
        DO 40 J = 1,3
          Q(I,J) = 0.0
          DO 35 K = 1,3
            Q(I,J) = Q(I,J) + TI(I,K)*P(K,J)
35    CONTINUE
40    CONTINUE
45 CONTINUE
      DENOM = 0.0
      DO 55 I = 1,3
        DO 50 J = 1,3
          DENOM = DENOM + AL(I)*Q(I,J)
50    CONTINUE
55 CONTINUE
      CS(N) = NUMER/DENOM
      S = S+0.01
      IF(S.LE.L)GOTO 100
*
* FIND THE TIME THAT HAS MINIMUM COST
*
```



```

      K = 1
200 IF(K.EQ.N)THEN
      SMALL = K*0.01
      ELSEIF(CS(K+1).GT.CS(K))THEN
      SMALL = K*0.01
      ELSEIF(CS(K+1).LE.CS(K))THEN
      SMALL = (K+1)*0.01
      K = K + 1
      GOTO 200
    ENDIF
*
* OUTPUT OPTIMAL REPLACEMENT AGE
*
      OPEN(UNIT = 3,FILE = 'OPTAGE')
      WRITE(3,333)SMALL , PI
333 FORMAT(1X,F6.3,4X,E11.4)
      OPEN(UNIT = 4,FILE = 'OPTCOST')
      WRITE(4,*)CS(K)
555 STOP
      END

```

\*\*\*\*\*

```

      SUBROUTINE EXPON(A,EXPA,PI)
C
C THIS SUBROUTINE USES TO FIND THE EXPONENTIAL OF A 3X3 MATRIX.
C IT WORK WITH COUPLE OF IMSL SUBROUTINES
C ('EVCRG', 'LINCGR', 'EPIRG').
C
      INTEGER LDA,LDEVEC,N,NOUT,I,J,K
      PARAMETER(N=3,LDA=N,LDEVEC=N,LDUINV=N,LD=N,LDU=N)
      REAL A(LDA,N),PI,EXPA(3,3)
      COMPLEX EVAL(N), EVEC(LDEVEC,N), ELD(3,3), UINV(3,3),
      COMPLEX C(3,3), D(3,3)
      OPEN(UNIT=1,FILE='EXPONENT')
*
* CALCULATE THE EIGENVALUES AND EIGENVECTORS
*
      CALL EVCRG(N,A,LDA,EVAL,EVEC,LDEVEC)
      DO 15 I = 1,3
        DO 10 J = 1,3
          ELD(I,J) = 0.0
10    CONTINUE
15  CONTINUE

```

```

ELD(1,1) = CEXP(EVAL(1))
ELD(2,2) = CEXP(EVAL(2))
ELD(3,3) = CEXP(EVAL(3))
*
* CALCULATE THE INVERSE EIGENVECTERS' MATRIX
*
CALL LINCG(N,EVEC,LDU,UINV,LDUINV)
DO 30 I = 1,3
  DO 25 J = 1,3
    C(I,J) = 0.0
    DO 20 K = 1,3
      C(I,J) = C(I,J) + ELD(I,K)*UINV(K,J)
20    CONTINUE
25  CONTINUE
30 CONTINUE
  DO 45 I = 1,3
    DO 40 J = 1,3
      D(I,J) = 0.0
      DO 35 K = 1,3
        D(I,J) = D(I,J) + EVEC(I,K)*C(K,J)
35    CONTINUE
40  CONTINUE
45 CONTINUE
  DO 55 I = 1,3
    DO 50 J = 1,3
      EXPA(I,J) = REAL(D(I,J))
50  CONTINUE
55 CONTINUE
C
C CALCULATE THE PERFORMANCE INDEX OF EIGENVALUES AND
C EIGENVECTERS IF LESS THAN 1 'EXCELLENT',IF IT IS BETWEEN 1
C AND 100 'GOOD', IF IT IS GREATER THAN 100 'THE SOLUTION IS
C SUSPECTED'
C
PI = EPIRG(N,N,A,LDA,EVAL,EVEC,LDEVEC)
RETURN
END

```

# PROGRAM SIMLIF

```
C
C THIS PROGRAM PROVIDES A RANDOM NUMBER OF A PHASE TYPE
C DISTRIBUTION THAT HAVE SPECIFIC REPRESENTATION WITH AN
C INITIAL PROBABILITIES VECTOR(ALPHA) AND A MATRIX OF
C TRANSITION RATES BETWEEN TRANSIENT STATES. THIS PROGRAMS
C WORKS WITH SUBROUTINE 'RANNUM' AND MORE SPECIFIC CASE
C ONLY 4 STATES DISTRIBUTION. THE VARIABLES ARE AS FOLLOWS:
C   LD(I,J)= TRANSITION RATE FROM STATE I TO STATE J
C   P(I,J) = TRANSITION PROBABILITY FROM STATE I TO STATE J
C   AL(I) = INITIAL PROBABILITY OF BEING IN STATE I
C   ELIF = THE SUPPOSED UPPER BOUND EXPECTED LIFETIME
*
* INITIALIZATION
*
  INTEGER S,SEED,N,N1
  REAL U,EXP,LIF,ELIF
  REAL LD12,LD13,LD14,LD21,LD23,LD24,LD31,LD32,LD34
  REAL P12,P13,P21,P23,P31,P32,AL1,AL2,AL3,D
  DATA LD12,LD13,LD14,LD21,LD23,LD24,LD31,LD32,LD34
+   /5.0,5.0,5.0,6.0,6.0,6.0,7.0,7.0,7.0/
  DATA P12,P13,P21,P23,P31,P32,AL1,AL2,AL3,ELIF
+   /0.8,0.2,0.167,0.833,0.143,0.286,1.0,0.0,0.0,10.0/
  CALL EXCMS('FILEDEF 10 DISK FILE LFTEST (DISP MOD')
  OPEN(UNIT = 1,FILE ='LIFETIME')
  OPEN(UNIT = 2,FILE ='SEED')
  READ(2,*)SEED
  CLOSE(2)
  LIF = 0.0
*
* GENERATE UNIFORM 0 AND 1 FOR INITIAL PROBABILITY
*
  CALL RANNUM(1,SEED,0.0,1.0,0,U)
  IF (U.LE.AL1) THEN
    S = 1
  ELSEIF(U.LE.AL1+AL2)THEN
    S = 2
  ELSE
    S = 3
  ENDIF
*
* SIMULATE LIFETIME WHEN IS IN STATE 1
*
```

5 CONTINUE

```
IF((S.EQ.1) .AND. (LIF.LT.ELIF)) THEN
  CALL RANNUM(1,SEED,0.0,1.0,0,U)
  IF(U.GT.P12+P13)THEN
    S = 4
    CALL RANNUM(3,SEED,LD14,0,0,EXP)
    LIF = LIF + EXP
  ELSEIF(U.GT.P12)THEN
    S = 3
    CALL RANNUM(3,SEED,LD13,0,0,EXP)
    LIF = LIF + EXP
  ELSE
    S = 2
    CALL RANNUM(3,SEED,LD12,0,0,EXP)
    LIF = LIF + EXP
  ENDIF
ENDIF
```

\*

\* SIMULATE LIFETIME WHEN IS IN STATE 2

\*

```
IF((S.EQ.2) .AND. (LIF.LT.ELIF))THEN
  CALL RANNUM(1,SEED,0.0,1.0,0,U)
  IF(U.GT.P21+P23)THEN
    S = 4
    CALL RANNUM(3,SEED,LD24,0,0,EXP)
    LIF = LIF + EXP
  ELSEIF(U.GT.P21)THEN
    S = 3
    CALL RANNUM(3,SEED,LD23,0,0,EXP)
    LIF = LIF + EXP
  ELSE
    S = 1
    CALL RANNUM(3,SEED,LD21,0,0,EXP)
    LIF = LIF + EXP
  ENDIF
ENDIF
```

\*

\* SIMULATION LIFETIME WHEN IS IN STATE 3

\*

```
IF((S.EQ.3) .AND. (LIF.LT.ELIF))THEN
  CALL RANNUM(1,SEED,0.0,1.0,0,U)
  IF(U.GT.P31+P32)THEN
    S = 4
```

```

      CALL RANNUM(3,SEED,LD34,0,0,EXP)
      LIF = LIF + EXP
    ELSEIF(U.GT.P31)THEN
      S = 2
      CALL RANNUM(3,SEED,LD32,0,0,EXP)
      LIF = LIF + EXP
    ELSE
      S = 1
      CALL RANNUM(3,SEED,LD31,0,0,EXP)
      LIF = LIF + EXP
    ENDIF
  ENDIF
  IF((S.LT.4) .AND. (LIF.LT.ELIF))GOTO 5

```

\*

\* OUTPUT TO 'FILE LIFETIME A'

\*

```

      WRITE(10,*)LIF
      WRITE(1,*)LIF
      OPEN(UNIT = 2,FILE = 'SEED')
      WRITE(2,*)SEED
      STOP
      END

```

\*\*\*\*\*

SUBROUTINE RANNUM(DISTN,SEED,RPARM1,RPARM2,IPARM,X)

```

C
C  THIS SUBROUTINE PROVIDES A RANDOM NUMBER OF 7 DISTRIBUTION.
C  IT INTERFACES WITH THE LLRANDOMII ROUTINES PROVIDED IN THE
C  NONIMSL LIBRARY. THE PARAMETER AND CALLING PROCEDURE ARE
C  AS FOLLOWS:
C  DIST = DISTRIBUTION TYPE YOU WANT TO SELECT AN INTEGER
C         BETWEEN 1 AND 7.
C  SEED = THE RANDOM NUMBER SEED YOU WISH TO USE.
C  RPARM1, RPARM2, AND IPARM ARE REAL AND INTEGER PARAMETERS.
C  PASSED TO THE ROUTINE WITH MEANINGS WHICH VARY WITH THE
C  TYPE OF DISTRIBUTION YOU DESIRE.
C  X = THE RETURNED RANDOM NUMBER, IT IS ALWAYS REAL.
C  DISTRIBUTION NUMBERS AND THE ASSOCIATED PARM DEFINITIONS
C  1 = UNIFORM ON THE INTERVAL RPARM1 TO RPARM2
C  2 = NORMAL WITH MEAN RPARM1 AND VARIANCE RPARM2
C  3 = EXPONENTIAL WITH RATE RPARM1
C  4 = COUCHY WITH A = RPARM1 AND B = RPARM2
C  5 = GAMMA WITH SHAPE RPARM2 AND RATE RPARM1

```



```

C 6 = POISSON WITH RATE RPARAM1
C 7 = GEOMETRIC WITH P = RPARAM1
C
  REAL RPARAM1,RPARAM2,X
  INTEGER DISTN,SEED,IPARM,N
  REAL TEMP,VARIAT(1)
  IF(DISTN.LE.0 .OR. DISTN.GE.8)THEN
    WRITE(10,*)'DISTN DOES NOT EXIT'
    STOP
  ENDIF
*
  GOTO (10,20,30,40,50,60,70),DISTN
*
* GENERATE A UNIFORM BETWEEN RPARAM1 AND RPARAM2
*
10 CONTINUE
  IF(RPARAM1-RPARAM2.EQ.0)THEN
    WRITE(10,*)'ILLEGAL EQUAL RPARAMS IN RANNUM'
    STOP
  ENDIF
  IF(RPARAM1.GT.RPARAM2)THEN
    TEMP = RPARAM1
    RPARAM1 = RPARAM2
    RPARAM2 = TEMP
  ENDIF
  CALL LRND(SEED,VARIAT,1,1,0)
  VARIAT(1) = RPARAM1 + (RPARAM2-RPARAM1)*VARIAT(1)
  GOTO 99
*
* GENERATE A NORMAL WITH MEAN RPARAM1 AND STD. DEV. RPARAM2
*
20 CALL LNORM(SEED,VARIAT,1,1,0)
  WRITE(10,*)'NORMAL(0,1) ',VARIAT(1)
  VARIAT(1) = (VARIAT(1)*RPARAM2) + RPARAM1
  GOTO 99
*
* GENERATE AN EXPONENTIAL WITH RATE RPARAM1 (1/MEAN)
*
30 CONTINUE
  IF(RPARAM1.EQ.0.0)THEN
    WRITE(10,*)'ILLEGAL ZERO RATE IN RANNUM'
    STOP
  ENDIF

```

```

CALL LEXPN(SEED,VARIAT,1,1,0)
VARIAT(1) = VARIAT(1)/RPARM1
GOTO 99
*
* GENERATE A COUCHY WITH A = RPARM1 AND B = RPARM2
*
40 CONTINUE
  IF(RPARM2.LE.0.0)THEN
    WRITE(10,*)'ILLEGAL COUCHY SPREAD IN RANNUM, B= ',RPARM2
    STOP
  ENDIF
  CALL LCCHY(SEED,VARIAT,1,1,0)
  VARIAT(1) = (VARIAT(1)*RPARM2) + RPARM1
  GOTO 99
*
* GENERATE A GAMMA WITH SHAPE RPARM2 AND RATE RPARM1
*
50 CONTINUE
  IF(RPARM1.LE.0.0)THEN
    WRITE(10,*)'ILLEGAL NONPOSITIVE GAMMA RATE IN RANNUM'
    STOP
  ENDIF
  IF(RPARM2.LE.0.0)THEN
    WRITE(10,*)'ILLEGAL SHAPE PARAMETER IN RANNUM'
    STOP
  ENDIF
  CALL LGAMA(SEED,VARIAT,1,1,0,RPARM2)
  VARIAT(1) = VARIAT(1)*(1.0/RPARM1)
  GOTO 99
*
* GENERATE A POISSON WITH RATE RPARM1
*
60 CONTINUE
  IF(RPARM1.LE.0.0)THEN
    WRITE(10,*)'ILLEGAL POISSON RATE IN RANNUM'
    STOP
  ENDIF
  CALL LPOIS(SEED,VARIAT,1,1,0,RPARM1)
  GOTO 99
*
* GENERATE A GEOMETRIC WITH P = RPARM1
*
70 CONTINUE

```

```
IF(RPARAM1.LE.0.0)THEN  
  WRITE(10,*)'ILLEGAL GEOM. PROB. IN RANNUM'  
  STOP  
ENDIF  
CALL LGEOM(SEED,VARIAT,1,1,0,RPARAM1)  
GOTO 99
```

\*

```
99 CONTINUE  
  X = VARIAT(1)  
  RETURN  
END
```

## PROGRAM MOMENT

C  
C THIS PROGRAM USES TO COMPUTE THE SECOND AND THIRD  
C STANDARDIZED MOMENTS THAT SUPPORTS MOMENT MATCHING IN  
C GAMS PROGRAM.

\*

\* INITIALIZE DATA

\*

```
INTEGER LIMIT,N,N1,I,J,COUNT
REAL LIF, NEW(2), REPAGE, A, B, C, EX, EXSQ, EXC, EX3
REAL STD, SECOND, THIRD, C1, C2, AAC, OBJ
REAL RAWDAT(1000,2), FB(1000), P(1000), TOT, TCOST
DATA C1,C2,TOT,TCOST,AAC/100.0,500.0,0.0,0.0,0.0/
OPEN(UNIT = 1,FILE = 'LIFETIME',STATUS = 'OLD')
OPEN(UNIT = 2,FILE = 'OPTAGE',STATUS = 'OLD')
OPEN(UNIT = 3,FILE = 'RAWDATA',STATUS = 'OLD')
OPEN(UNIT = 4,FILE = 'MOMENT')
OPEN(UNIT = 5,FILE = 'COUNT',STATUS = 'OLD')
READ(1,*)LIF
READ(2,*)REPAGE
READ(5,*)COUNT
PRINT*, 'COUNT=',COUNT
N1 = 25 + COUNT
DO 5 I = 1,1000
    P(I) = 0.0
    FB(I) = 0.0
5 CONTINUE
DO 44 I = 1,N1-1
    READ(3,11)RAWDAT(I,1),RAWDAT(I,2)
11  FORMAT(1X,F7.4,4X,F3.1)
44 CONTINUE
CLOSE(3)
OPEN(UNIT=3,FILE='RAWDATA')
```

\*

\* CALCULATION

\*

```
IF(LIF.LE.REPAGE)THEN
    NEW(1) = LIF
    NEW(2) = 1.0
ELSE
    NEW(1) = REPAGE
    NEW(2) = 0.0
ENDIF
```

```

CALL INSERT(N1,NEW,RAWDAT)
DO 55 I = 1,N1
    WRITE(3,33) RAWDAT(I,1),RAWDAT(I,2)
33    FORMAT(1X,F7.4,4X,F3.1)
55 CONTINUE
    IF(RAWDAT(1,2) .NE. 0.0)THEN
        FB(1) = (N1-1.0)/N1
    ELSE
        FB(1) = 1.0
    ENDIF
    DO 10 I = 2,N1
        IF(RAWDAT(I,2).EQ.0.0)THEN
            FB(I) = FB(I-1)
        ELSEIF(I.LT.N1)THEN
            FB(I) = ((N1-I)/(N1+1.0-I))*FB(I-1)
        ELSE
            FB(N1) = 0.0
        ENDIF
    10 CONTINUE
*
    P(1) = 1.0-FB(1)
    DO 22 I = 2,N1
        P(I) = FB(I-1)-FB(I)
    22 CONTINUE
    EX = 0.0
    EXSQ = 0.0
    EXC = 0.0
    EX3 = 0.0
    DO 15 J = 1,N1
        TOT = TOT + RAWDAT(J,1)
        TCOST = TCOST + C2*RAWDAT(J,2) + C1*(1-RAWDAT(J,2))
        A = RAWDAT(J,1)*P(J)
        B = (RAWDAT(J,1)**2)*P(J)
        C = (RAWDAT(J,1)**3)*P(J)
        EX = EX + A
        EXSQ = EXSQ + B
        EXC = EXC + C
    15 CONTINUE
    AAC = TCOST/TOT
    STD = (EXSQ - EX**2)**0.5
    EX3 = EXC + (2*EX**3) - (3*EX*EXSQ)
    SECOND = STD/EX
    THIRD = EX3/(STD**3)

```



```

*
* OUTPUT
*
      WRITE(4,100)SECOND,THIRD,AAC,EX
100 FORMAT(2X,F11.8,4X,F11.8,4X,F8.3,4X,F6.3)
*   CLOSE(5)
*   OPEN(UNIT = 5,FILE = 'COUNT',STATUS = 'OLD')
*   WRITE(5,*)COUNT+1
      STOP
      END

```

```

*****
      SUBROUTINE INSERT(LIMIT,NEW,X)
C   THIS SUBROUTINE USES TO INSERT A PAIR 'NEW' DATA INTO THE
C   DATA FILE 'X' IN ASCENDING ORDER.
      INTEGER LIMIT,J,K
      REAL NEW(2),X(1000,2)
      LOGICAL DONE
      DONE = .FALSE.
      J = 1
5   IF((J.LT.LIMIT).AND.(.NOT.DONE))THEN
      IF(X(J,1).LT.NEW(1))THEN
          J =J+1
      ELSE
          DONE = .TRUE.
      ENDIF
      GOTO 5
    ENDIF
    IF(J.EQ.LIMIT)THEN
        X(J,1) = NEW(1)
        X(J,2) = NEW(2)
    ELSE
        IF(J.LT.LIMIT)THEN
            DO 10 K = LIMIT,J+1,-1
                X(K,1) = X(K-1,1)
                X(K,2) = X(K-1,2)
10        CONTINUE
            X(J,1) = NEW(1)
            X(J,2) = NEW(2)
        ENDIF
    ENDIF
    RETURN
    END

```

## PROGRAM DATA

C  
C THIS PROGRAM PROVIDES THE RESULT SEQUENTIAL ESTIMATED  
C ENTRIES TRANSITION RATES  $T(I,J)$  AND INITIAL STATE PROB.  
C ALPAH(I),AND ALSO ADJUST, PREPARE ALL THE INITIAL DATA THAT  
C WILL BE USED FOR THE NEXT SEQUENTIAL ESTIMATION.  
C THE VARIABLE ARE AS FOLLOWS:  
C A,B,C  
C D,E,F = THE ESTIMATED TRANSITION MATRIX  
C G,H,I  
C AL1,AL2,AL3 = THE ESTIMATED INITIAL STATES PROBABILITIES  
C X = RANDOM VARIABLE OF LIFETIME  
C OPTAGE = OPTIMAL AGE REPLACEMENT  
C SECOND = THE SECOND STANDARDIZED MOMENT  
C THIRD = THE THIRD STANDARDIZED MOMENT  
C OBJ = OBJECTIVE FUNCTION VALUE OF GAMS PROGRAM

\*

\* INPUT

\*

### INTEGER COUNT

REAL A,B,C,D,E,F,G,H,I,AL1,AL2,AL3,X  
REAL EX,OPTCS,OPTAGE,Z,SECOND,THIRD,DI,OBJ,AAC,PI  
OPEN(UNIT = 10,FILE = 'OBJ',STATUS = 'OLD')  
OPEN(UNIT = 11,FILE = 'ALPHA',STATUS = 'OLD')  
OPEN(UNIT = 12,FILE = 'BRAVO',STATUS = 'OLD')  
OPEN(UNIT = 13,FILE = 'CHAREE',STATUS = 'OLD')  
OPEN(UNIT = 14,FILE = 'DELTA',STATUS = 'OLD')  
OPEN(UNIT = 15,FILE = 'ECHO',STATUS = 'OLD')  
OPEN(UNIT = 16,FILE = 'FOXTROT',STATUS = 'OLD')  
OPEN(UNIT = 17,FILE = 'GOLF',STATUS = 'OLD')  
OPEN(UNIT = 18,FILE = 'HOTEL',STATUS = 'OLD')  
OPEN(UNIT = 19,FILE = 'INDIA',STATUS = 'OLD')  
OPEN(UNIT = 20,FILE = 'AL1',STATUS = 'OLD')  
OPEN(UNIT = 21,FILE = 'AL2',STATUS = 'OLD')  
OPEN(UNIT = 22,FILE = 'AL3',STATUS = 'OLD')  
OPEN(UNIT = 23,FILE = 'OPTAGE',STATUS = 'OLD')  
OPEN(UNIT = 24,FILE = 'LIFETIME',STATUS = 'OLD')  
OPEN(UNIT = 25,FILE = 'MOMENT',STATUS = 'OLD')  
OPEN(UNIT = 26,FILE = 'COUNT',STATUS = 'OLD')  
OPEN(UNIT = 27,FILE = 'OPTCOST',STATUS = 'OLD')  
CALL EXCMS('FILEDEF 50 DISK FILE ESTDAT1 (DISP MOD)')  
CALL EXCMS('FILEDEF 60 DISK FILE ESTDAT2 (DISP MOD)')  
CALL EXCMS('FILEDEF 70 DISK FILE ESTDAT3 (DISP MOD)')

```

OPEN(UNIT = 30,FILE = 'SCALAR2')
READ(10,*)OBJ
IF(OBJ .GT. 0.05)THEN
  A = -1.0
  B = 0.3
  C = 0.3
  D = 0.3
  E = -1.0
  F = 0.3
  G = 0.3
  H = 0.3
  I = -1.0
ELSE
  READ(11,*)A
  READ(12,*)B
  READ(13,*)C
  READ(14,*)D
  READ(15,*)E
  READ(16,*)F
  READ(17,*)G
  READ(18,*)H
  READ(19,*)I
ENDIF
READ(20,*)AL1
READ(21,*)AL2
READ(22,*)AL3
READ(23,*)OPTAGE,PI
READ(24,*)X
READ(25,*)SECOND,THIRD,AAC,EX
READ(26,*)COUNT
READ(27,*)OPTCS
CLOSE(26)
PRINT*, 'OBJ=', OBJ

```

\*

```

IF(OPTAGE .GT. X)THEN
  Z = X
  DI = 1.0
ELSE
  Z = OPTAGE
  DI = 0.0
ENDIF

```

```

*
* OUTPUT
*
OPEN(UNIT = 26,FILE ='COUNT',STATUS ='OLD')
WRITE(26,*)COUNT+1
WRITE(50,200)COUNT,A,B,C,D,E,F,G,H,I
WRITE(60,300)COUNT,OPTCS,OPTAGE,X,Z,DI,AAC
WRITE(70,100)COUNT,AL1,AL2,AL3,SECOND,THIRD,OBJ,PI

*
* PREPARE THE DATA FOR "GAMS" PROGRAM
*
WRITE(30,*)' SCALAR'
WRITE(30,400)' AA INITIAL SOLUTION OF A /',A,' /'
WRITE(30,400)' BB INITIAL SOLUTION OF B /',B,' /'
WRITE(30,400)' CC INITIAL SOLUTION OF C /',C,' /'
WRITE(30,400)' DD INITIAL SOLUTION OF D /',D,' /'
WRITE(30,400)' EE INITIAL SOLUTION OF A /',E,' /'
WRITE(30,400)' FF INITIAL SOLUTION OF B /',F,' /'
WRITE(30,400)' GG INITIAL SOLUTION OF C /',G,' /'
WRITE(30,400)' HH INITIAL SOLUTION OF D /',H,' /'
WRITE(30,400)' II INITIAL SOLUTION OF D /',I,' /'
WRITE(30,500)' ALP1 INITIAL SOLUTION OF AL1 /',AL1,' /'
WRITE(30,500)' ALP2 INITIAL SOLUTION OF AL2 /',AL2,' /'
WRITE(30,500)' ALP3 INITIAL SOLUTION OF AL3 /',AL3,' /'
WRITE(30,600)' MM2 SECOND STANDARD MOMENT /',SECOND,' /'
WRITE(30,600)' MM3 THIRD STANDARD MOMENT /',THIRD,' /'
WRITE(30,700)' EX EXPECTED VALUE OF LIFETIME /',EX,' /';'
100 FORMAT(1X,I3,3(2X,F5.2),2(2X,F8.5),2X,F4.2,2X,E11.4)
200 FORMAT(1X,I3,2X,9(2X,F6.2))
300 FORMAT(1X,I3,2X,F8.3,2X,F5.2,2(4X,F7.4),4X,F3.1,4X,F7.3)
400 FORMAT(A35,F6.2,A2)
500 FORMAT(A39,F5.2,A2)
600 FORMAT(A36,F11.8,A1)
700 FORMAT(A40,F6.3,A4)
STOP
END

```

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